Ohno Conjecture on the Zeta Functions associated with the Space of Binary Cubic Forms

Jin Nakagawa
Joetsu Univ. Edu. Japan

1 Introduction

Let $\Gamma = \text{SL}_2(\mathbb{Z})$ and let

$$x(u, v) = x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3$$

be a binary cubic form with int. coeff.

The action of a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is defined by

$$(\gamma x)(u, v) = x(au + cv, bu + dv).$$

The discriminant of $x$ is defined by

$$D(x) = 18 x_1 x_2 x_3 x_4 + x_2^2 x_3^2 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2.$$ 

Then

$$D(\gamma x) = D(x), \quad \forall \gamma \in \Gamma.$$ 

Let

$$L = \{x(u, v); \ x_i \in \mathbb{Z}\},$$
$$\hat{L} = \{x \in L; x_2, x_3 \in 3\mathbb{Z}\}.$$ 

These sets are $\Gamma$-inv.

For any $n \in \mathbb{Z}$, $n \neq 0$, let

$$L(n) = \{x \in L; D(x) = n\},$$
$$\hat{L}(n) = \{x \in \hat{L}; D(x) = n\}.$$
We define the class numbers

\[ h(n) = \#(\Gamma \backslash L(n)), \]
\[ \hat{h}(n) = \#(\Gamma \backslash \hat{L}(n)). \]

Eisenstein, Arndt, Hermite, 19C

\[ h(n) < \infty, \quad \text{Tables} \]

To be more precise, let

\[ \Gamma_x = \{ \gamma \in \Gamma; \gamma x = x \}. \]

Then

\[ |\Gamma_x| = \begin{cases} 1 \text{ or } 3, & D(x) > 0, \\ 1, & D(x) < 0. \end{cases} \]

According to the order of the isotropy subgroup, we define

\[ h_1(n) = \#(\Gamma \backslash \{ x \in L(n); |\Gamma_x| = 1 \}), \]
\[ h_2(n) = \#(\Gamma \backslash \{ x \in L(n); |\Gamma_x| = 3 \}). \]

We define \( \hat{h}_1(n) \) and \( \hat{h}_2(n) \) similarly.

Shintani, 1972.

\[ \xi_1(L, s) = \sum_{n=1}^{\infty} \frac{h_1(n) + 3^{-1}h_2(n)}{n^s}, \]
\[ \xi_2(L, s) = \sum_{n=1}^{\infty} \frac{h(-n)}{n^s}, \]
\[ \xi_1(\hat{L}, s) = \sum_{n=1}^{\infty} \frac{\hat{h}_1(n) + 3^{-1}\hat{h}_2(n)}{n^s}, \]
\[ \xi_2(\hat{L}, s) = \sum_{n=1}^{\infty} \frac{\hat{h}(-n)}{n^s}. \]
These Dirichlet series are abs. conv. for $\Re s > 1$, cont. to mero. func. on $\mathbb{C}$, only poles at $s = 1, \frac{5}{6}$ (simple), satisfy the func. eq.

$$
\begin{pmatrix}
\xi_1(L, 1 - s) \\
\xi_2(L, 1 - s)
\end{pmatrix}
= \Gamma\left(s - \frac{1}{6}\right) \Gamma(s) 2 \Gamma\left(s + \frac{1}{6}\right)
\times 2^{-1} \Gamma_6 - 2 \pi^{-4s}
\begin{pmatrix}
\sin 2\pi s & \sin \pi s \\
3 \sin \pi s & \sin 2\pi s
\end{pmatrix}
\begin{pmatrix}
\xi_1(\hat{L}, s) \\
\xi_2(\hat{L}, s)
\end{pmatrix}
$$


(i) $\xi_1(\hat{L}, s) = 3^{-3s} \xi_2(L, s)$,

(ii) $\xi_2(\hat{L}, s) = 3^{1-3s} \xi_1(L, s)$.

We can rewrite the conjecture into the following relations of class numbers.

(i) $\hat{h}_1(27n) + \frac{1}{3} \hat{h}_2(27n) = h(-n)$ $\forall n > 0$;

(ii) $\hat{h}(-27n) = 3h_1(n) + h_2(n)$ $\forall n > 0$.

The func. eq. implies (i) $\iff$ (ii).

He also showed that under the conjecture, Diagonalization of func. eq. by Datskovsky–Wright implies simpler and more symmetric func. eqv of a single zeta function:

$$Z_\pm(1 - s) = Z_\pm(s),$$

where

$$Z_\pm(s) = 2^s 3^\frac{\pm s}{2} \pi^{-2s}
\times \Gamma(s) \Gamma\left(\frac{s}{2} + \frac{1}{4} \mp \frac{1}{6}\right) \Gamma\left(\frac{s}{2} + \frac{1}{4} \pm \frac{1}{3}\right)
\times \left(3^\pm \xi_1(L, s) \mp \xi_2(L, s)\right).$$
For simplicity, denote by $\tilde{h}(27n)$ the left hand side of (i)’:

$$\tilde{h}(27n) = \hat{h}_1(27n) + \frac{1}{3}\hat{h}_2(27n).$$

To prove the conjecture, it is enough to show

$$\hat{h}(27n) = h(-n) \quad \forall n > 0.$$ 

By proving this equation directly, I succeeded to prove the conjecture.

**Theorem 1.** The conjecture is true.

### 2 Outline of the proof

Let $x \in \hat{L}(27n)$. We write

$$x(u, v) = x_1 u^3 + 3x_2 u^2 v + 3x_3 u v^2 + x_4 v^3, \quad x_i \in \mathbb{Z}.$$ 

Let $H_x$ be the Hessian of $x$.

$$H_x(u, v) = -\frac{1}{36} \begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 x}{\partial u \partial v} \\ \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 x}{\partial v^2} \end{vmatrix}.$$ 

Then $H_x$ is a positive definite integral binary quadratic form with disc. $-n$, and

$$H_{\gamma x} = \gamma H_x \quad (\forall \gamma \in \Gamma).$$

Let $k = \mathbb{Q}(\sqrt{-n})$. We now assume that $-n$ is a fund. disc., i.e. the disc. of $k$. 

4
\[ \Gamma \backslash \{ \text{bin. quad. forms with disc} \ - \ n \} \longleftrightarrow \ Cl_k \]

\[ \bigcup \bigcup \]

\[ \Gamma \backslash \{ H_x; x \in \hat{L}(27n) \} \longleftrightarrow Cl_k^{(3)} \]

\[ Cl_k^{(3)} = \{ c \in Cl_k; c^3 = 1 \}. \]

Hence

\[ \tilde{h}(27n) = |Cl_k^{(3)}|. \]

Datskovsky–Wright, 1986

\[ \frac{1}{2} \zeta_2(L, s) = \sum_{\kappa: \text{cubic f., } D_K < 0} |D_K|^{-s} \eta_K(2s) + \frac{1}{2} \sum_{k: \text{imag. quad. f.}} |D_k|^{-s} \eta_{\mathbb{Q}(k)}(2s), \]

where

\[ \eta_A(s) = \zeta(2s)\zeta(3s - 1) \frac{\zeta_A(s)}{\zeta_A(2s)}, \]

\[ \zeta_A(s) = \prod_i \zeta_{K_i}(s), \quad A = \oplus_i K_i. \]

This expression implies that

\[ h(-n) = 2\#\{ \text{cubic fields with disc.} \ - \ n \} + 1 = |Cl_k^{(3)}| \]

Thus we have

\[ \tilde{h}(27n) = h(-n) \]

under the assumption that \(-n\) is a fund. disc..

The case \(-n = m^2D_k, m: \text{square free}\), is proved by generalizing the argument above. The case of arbitrary \(m\) is proved by some recursive formulae for \(h(-np^{2r})\) and \(\tilde{h}(27np^{2r}), r = 0, 1, 2, \ldots\) coming from D-W’s expression.
3 Application

Let $N_3(n)$ be the number of the cubic fields with discriminant $n$.

**Theorem 2.** Let $k$ be an imaginary quadratic field with $k \neq \mathbb{Q}(\sqrt{-3})$ and put $n = |D_k|$. If $3 \nmid n$, then

$$N_3(3n) + N_3(27n) = N_3(-n),$$

$$N_3(-n) + N_3(-81n) = 3N_3(3n) + 1.$$  

If $3|n$, then

$$N_3(n/3) + N_3(27n) = N_3(-n),$$

$$N_3(-n) + N_3(-9n) = 3N_3(n/3) + 1.$$  

For any quadratic field $k$ and for any positive integer $c$, denote by $O_{k,c}$ the order of $k$ of conductor $c$, and denote by $r_{k,c}$ the 3-rank of the ideal class group of $O_{k,c}$. By class field theory, Theorem 2 is equivalent to the following

**Theorem 3.** Let $k$ and $n$ be as in Theorem 2 and let $k'$ be the real quadratic field $\mathbb{Q}(\sqrt{3n})$. If $3 \nmid n$, then $r_{k',3} = r_{k,1}$ and $r_{k,9} = r_{k',1} + 1$. If $3|n$, then $r_{k',9} = r_{k,1}$ and $r_{k,3} = r_{k',1} + 1$.

**Remark 4.** Theorem 3 can be viewed as a precise version of Scholz’s reflection theorem.

**Remark 5.** The residue of $\xi_2(L, s)$ at $s = \frac{5}{6}$ is equal to that of $\sqrt{3}\xi_1(L, s)$. Hence $Z_-(s)$ has only one pole at $s = 1$, while $Z_+(s)$ has exactly two poles at $s = 1$ and $s = \frac{5}{6}$.

**Remark 6.** If the direct bijection between classes in question can be easily described in some way, it should be very interesting. However, I have no idea on this. In general, the number of the equivalence classes of irreducible forms in $\hat{L}(27n)$ does not coincide with that of irreducible forms in $L(-n)$. 
