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Teaching Ratio, Proportion and Continued-Proportion between Physical Quantities

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Abstract

This paper describes how to bridge a gap between number and quantity ratios in science education. The concept of concentration of solution can be interpreted as a continued-proportion between mass and volume, because it looks like a class of ratios between abstract numbers in mathematics. This interpretation of concentration makes it easy for students to solve diverse problems relating to concentration. Such an approach is also applicable to solving physical and/or chemical problems concerning density and molar mass.

Keywords: Ratio, proportion, continued-proportion, quantity calculus, class, representative, physical quantity for equating, concentration, density, molar mass, SI, science education.

1. Introduction

The origin of the concept of ratio between numbers can be traced back to the dawn of mathematics,[1] but the ratio between physical quantities such as weight (water) per cup (volume) was used long before that. This ratio of one physical quantity to another is so fundamental and so widely accepted in modern science and also every-day life that it needs to be taught in science education.

Ratios and proportions between numbers obey the arithmetical rules (addition, subtraction, multiplication and division) of algebra.[2] There are no such simple rules for ratio and proportion between physical quantities. Students know what the number ratio means and how it works under the arithmetical rules, and science teachers[3] make use of part of the number ratio method for quantity calculus in natural science. The two methods however, as shown below, are not entirely replaceable. This paper analyzes the proportion between physical quantities in terms of two axioms that the quantity ratios obey; this processing bridges the gap between the two methods.

A set $\{x_1, x_2, x_3, \ldots\}$ of numbers x_j is given, and each of x_j is identified with an abstract number x; namely, $x = \{x_1, x_2, x_3, \ldots\}$. Then x and each of x_j, respectively, are

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called a class and a representative for the set of x_j in mathematics; and if each of x_j is a ratio between numbers, the set is called a continued-proportion or a continuous-proportion.[2] The present contribution treats the concentration c of a salt solution such that c is a class of c_j, where each of c_j is a quantity ratio for a portion of the solution; *i.e.*, $c_1 = c_2 = c_3 = \ldots$ is a continued-proportion between mass (of salt) and volume. This interpretation of concentration helps students to solve diverse problems relating to concentration. This approach is also applicable to the solving of many physical and/or chemical problems concerning density (mass per volume) and molar mass (mass per mole).

2. Ratio and a new physical quantity

Quantity calculus, namely, a method of handling physical quantities,[4] is the background of the International System of Units (SI). This method describes the value of a physical quantity as the product of a numerical value and a unit;

$$\text{physical quantity} = \text{numerical value} \times \text{unit}$$

This equation is hereafter written as $q = a \times \mathrm{u}$. Here q and u are physical quantities of the same kind. This type of representation was established in the late nineteenth century.[5] Measuring q compares q with u, so as to obtain the ratio between them; $q/\mathrm{u} = a$ (numerical value), and hence $q = a \times \mathrm{u}$.

Forming a ratio between physical quantities is a useful way to compare one with the other in science and technology. Let q_1 and q_2 be two physical quantities; $q_1 = a_1\mathrm{u}_1$ and $q_2 = a_2\mathrm{u}_2$ are given. The ratio $q_1 : q_2$ is equivalent to the quotient q_1/q_2, which leads to $q_1/q_2 = a_3\mathrm{u}_3 = q_3$, where $a_3 = a_1/a_2$ (a numerical value) and $\mathrm{u}_3 = \mathrm{u}_1/\mathrm{u}_2$ (a unit). In other words, the ratio of one physical quantity to another is a new physical quantity. One says that the numerator q_1 and the denominator q_2 have the same dimensions when $\mathrm{u}_3 = 1$ or when $q_1/q_2 = a_3$, and that the dimensions of q_1 and q_2 are different when q_1/q_2 is not a numerical value.[6]

Let us consider the meaning of $q_1/q_2 = q_3$ for an aqueous solution as an example. Suppose the following situation: A teacher takes a sample (volume v_1) from a salt solution, and determines the mass (m_1) of salt therein; say $v_1 = 10$ mL and $m_1 = 20$ mg. The teacher shows the students the calculation

$$m_1/v_1 = 20\ \mathrm{mg}/(10\ \mathrm{mL}) = 2\ \mathrm{g/L} = c_1$$

and says that 2 g/L $= c_1$ is the concentration with the measuring unit g/L. It should however be noted that the calculation neglects the gap between 20 mg/(10 mL) for a sample and 2 g/L for the whole solution. For a homogeneous solution one has m/v

$= c$; m is in direct ratio to v, and c is the proportionality constant. It is possible to equate the two ratios, 20 mg/(10 mL) and 2 g/L, given the condition that the solution in question be homogeneous. The formalized equation $m/v = c$ has a two-fold meaning, the whole solution as in c and a portion of the solution as in m/v, which often confuses beginners in science.[7] Students could easily bridge the gap and then solve the following problems by considering the ratio for each portion of solution to be a representative of a class of ratios, *i.e.*, by interpreting concentration as a continued-proportion between mass and volume.

Three problems are given for student practice:

(i) What is the mass of salt in 10 mL of a 2 g/L solution? The answer[8] may be had by calculating the product (2 g/L) (10 mL).

(ii) What is the volume of a sample containing 20 mg of salt from a 2 g/L solution? It is rather difficult for students to do the quantity ratio (20 mg)/(2 g/L), mass divided by mass divided by volume.

(iii) What is the volume of a portion containing 20 mg of salt from 500 mL of a solution to obtain 1 g of salt? The first step is to make the ratio 1 g/(500 mL); and the second is to achieve the ratio (20 mg)/(1 g/(500 mL)). This procedure is more complicated than the previous one.

The three problems are solvable[9] by use of the formalized equation $m/v = c$. The unknown m for (i) is given by $m = c \times v$, and the unknown v for (ii-iii) is expressed as $v = m/c$.

An alternative[8] to the use of the formalized equation $m/v = c$ for problem-solving is to equate two ratios; one can obtain $m/(20 \text{ mL}) = 2 \text{ g/L}$ for (i), $20 \text{ mg}/v = 2 \text{ g/L}$ for (ii) and $20 \text{ mg}/v = 1 \text{ g}/(500 \text{ mL})$ for (iii). Each of these equations is just a proportion between mass and volume. The concept of proportion between physical quantities is discussed in the next section.

Ratios q_1/q_2 such as concentration, density and molar mass, of a physical system, are called *intensive* when they are independent of the size of the system.[10] Two solutions, 20 mg of salt in a 10 mL portion and 15 mg of salt in a 5 mL portion, are given. One wants to compare two systems of different size, and hence, one has to use the intensive property of solution or the ratio of mass to volume. The ratio between ratios is applicable to such problems; *i.e.*,

$$(15 \text{ mg}/5 \text{ mL})/(20 \text{ mg}/10 \text{ mL}) = 3/2$$

In the above-mentioned calculations of ratios it is tacitly assumed that physical quantities satisfy the following axioms, where $q/1$ is rewritten as q. One can judge by use of the axioms whether or not the multiplication and the division of ratios and proportions are valid in quantity calculus.

Axiom 1 (product of ratios): The product of two ratios q_1/q_2 and q_3/q_4 is a ratio; $(q_1/q_2)(q_3/q_4) = (q_1q_3)/(q_2q_4)$.
Axiom 2 (unit ratio): $(q_1/q_1)(q_2/q_3) = (q_1q_2)/(q_1q_3) = q_2/q_3$; $q_1/q_1 = 1$.

3. Proportion and equating two intensive quantities

An equality of two ratios q_1/q_2 and q_3/q_4 is called a proportion[2]; one can write it as $q_1/q_2 = q_3/q_4$ and in short as $q_{12} = q_{34}$, where $q_{12} = a_{12}\mathrm{u}_{12} = q_1/q_2$ and $q_{34} = a_{34}\mathrm{u}_{34} = q_3/q_4$. The equality in $q_{12} = q_{34}$ means two equalities; both sides of $q_{12} = q_{34}$ have dimensions of the same kind (*i.e.*, $\mathrm{u}_{12} = \mathrm{u}_{34}$) and the same numerical values (*i.e.*, $a_{12} = a_{34}$). The physical quantities for equating[8] the two ratios q_{12} and q_{34} are hereafter abbreviated to PQE.

It should be noted that two physical quantities in a proportion $q_1/q_2 = q_3/q_4$ do not necessarily have the same dimensions; for example, the unit of energy in SI is defined as $\mathrm{J} = \mathrm{N} \times \mathrm{m}$ (Joule = Newton × meter), from which it is derived that $\mathrm{J/m^3} = \mathrm{N/m^2}$ whose PQE is pressure. In addition, recall that the question of whether two physical quantities can be equated in a given physico-chemical situation has been considered. In fact, not all cases where $q_1/q_2 = a$ and $q_3/q_4 = a$ can be equated as $q_1/q_2 = q_3/q_4$. This situation is different from that for numbers.

Many problems in which two (or more) systems of different size are compared lead to proportion; for example, a salt solution and its portions in Problems (i-iii). The PQE for solving such problems must be intensive quantities because extensive ones for two or more systems can not be compared. One should become conscious of the PQE in order to solve Problems (i-iii). The proportions (PQE: mass per volume) for solving the latter problems have been shown in the previous section.

It is well known[2] that in every valid proportion between numbers the product of the inner terms is equal to that of the outer terms; that is, $x_1/x_2 = x_3/x_4$ for four real numbers x_1, x_2, x_3 and x_4 (not zero), implies that $x_1x_4 = x_2x_3$; and the interchange of the two inner terms, the two outer terms or the inner terms with the outer terms leads to another valid proportion, namely, $x_1/x_2 = x_3/x_4$ which implies that $x_1/x_3 = x_2/x_4$, $x_4/x_2 = x_3/x_1$ and $x_2/x_1 = x_4/x_3$. One may well ask, are these statements true for proportions between physical quantities? In fact, one can easily prove the following theorem in terms of Axioms 1 and 2.

Theorem of Proportions between (non-zero) physical quantities: One of the equations, $q_1/q_2 = q_3/q_4$, $q_1q_4 = q_2q_3$, $q_1/q_3 = q_2/q_4$, $q_4/q_2 = q_3/q_1$ and $q_2/q_1 = q_4/q_3$, implies another.

4. Continued-proportion and finding a class of physical quantities

A chain of proportions, $x_1/x_2 = x_3/x_4 = x_5/x_6 = \ldots$, for any associated numbers x_j and x_{j+1}, is called a continued-proportion[2] (or continuous proportion); an abbreviation for the chain is

$$x_1 : x_3 : x_5 : \ldots = x_2 : x_4 : x_6 : \ldots$$

In the previous section the ratio $m_1/v_1 = c_1$ for a portion of an aqueous solution was calculated. Repeating such an operation leads to a series $\{c_1, c_2, c_3, \ldots\}$ of data. Each of c_j is a new physical quantity. The teacher identifies all the c_j with c; namely, $c = \{c_1, c_2, c_3, \ldots\}$. In mathematics, c and each of c_j, respectively, are called a class and a representative for the set $\{c_1, c_2, c_3, \ldots\}$. In other words, one can interpret concentration as a class of m_j/v_j or as a continued-proportion between mass and volume.

A 2 g/L solution is provided for student practice (i); this situation means that a continued-proportion, for example,

$$2 \text{ g/L} = 2 \text{ mg/mL} = 20 \text{ mg/(10 mL)} = 2 \text{ kg/kL} = \ldots$$

is given, and that the value 2 g/L is a representative for the set

$$\{2 \text{ g/L}, 2 \text{ mg/mL}, 20 \text{ mg/(10 mL)}, 2 \text{ kg/kL}, \ldots\}$$

whose PQE is concentration. In solving Problems (i-iii) two ratios were equated with an unknown variable. Let m be the unknown mass in Problem (i); then the ratio of mass to volume for a 10 mL portion is expressible as $m/(10 \text{ mL})$, which is also a representative for the set of data. Hence, one has an equation[8] $m/(10 \text{ mL}) = 2$ g/L. Considering $m/(10 \text{ mL})$ and 2 g/L to be two representatives for a class of mass per volume makes it possible for students to equate the two ratios in solving Problem (i).

In classrooms, science teachers often say that the concentration of aqueous solutions is denoted by, for example, 2 g/L. They regard the form $c = 2$ g/L as being applicable to a single solution, while some students have only one image of a 1 L portion of solution. Neither of these cases has the intensive property of solution. The intensive property of concentration, as has been discussed, is just a continued-proportion. Students should be taught the expression 2 g/L in $c = 2$ g/L as a representative for concentration.

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