On Resolutions of Generalized Metric Spaces

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1.1 Introduction

In this chapter, all spaces are assumed to be a regular T_1-space. The letter $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote all natural numbers, rational numbers, real numbers, respectively. For a space X, let us denote the topology by $\tau(X)$ or τ. For a family $\mathcal{U}$ of subsets of a space X and for a subset A of X, we write

$$S(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\},$$

and for a point $p \in X$, we write

$$C(p,\mathcal{U}) = \bigcap\{U \in \mathcal{U} \mid p \in U\}.$$

For a subset A of X, we write the restriction of $\mathcal{U}$ to A by $\mathcal{U}|A$. In 1953, Nagata stated in [10] that Ceder space, stated below, has a σ-closure-preserving base. But he did not give the direct proof. In 1961 Ceder defined M_1-space as a generalization of metric spaces, and taking it into account, he give the direct proof that Ceder space is an M_1-space, i.e., it has a σ-closure-preserving base, [1]. But his proof is very complicated. So, in 1992, Watson showed that both Ceder space and McAuley space stated below, are obtained as special resolutions of the half plane at each point in $\mathbb{R} \times \{0\}$ into 2 point space by suitable continuous mappings, [12, Lemma 3.3.17] and proposed the problem to find a simple proof to Ceder's proof, [12, Problem 3.3.18].

In this chapter, we give an answer to this problem. The essential point of our proof is based on the fact that if $\mathcal{U}$ is a locally finite family of subsets of a space X, then the family $\{\bigcup\mathcal{U}_0 \mid \mathcal{U}_0 \subset \mathcal{U}\}$ is closure-preserving in X and use the fact that X_0 has a *uniformly approaching anti-cover* in both McAuley space and Ceder space. This concept is due to Nagami [8].

Finally, we determine the position of both spaces in the sequence from metric space through M_1-spaces, that is, we show that they are D-space in the sense of Nagami [8], which is weaker than Lašnev spaces.

As for M_1-spaces and other related terms, refer to Gruenhage [4].

1.2 McAuley space is an M_1-space

We state the topology of two spaces known already, one of which is called here McAuley space and the other Ceder space:

Definition 1.1. Let $X = X_0 \cup X_1$, where

$$X_0 = \{(x,0) \mid x \in \mathbb{R}\}, \quad X_1 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\},$$

and we topologize X by defining neighborhood bases at each point $p \in X$ by the following two ways: First, if (1) $\in X_1$, then p has a usual neighborhoodbase and if (2) $p = (x,0) \in X_0$, then p has a neighborhood base $\{M(p : 1/n) \mid n \in \mathbb{N}\}$, where

$$M\left(p : \frac{1}{n}\right) = \{p\} \cup \left\{(x',y') \in X \mid y' < \frac{1}{n}|x' - x| < \frac{1}{n^2}\right\}, \quad n \in \mathbb{N}.$$

Then the space X thus defined X is called *McAuley space.*

On the other hand, if (2) $p = (x,0)$ has a neighborhood base $\{C(p : 1/n) \mid n \in \mathbb{N}\}$, where

$$C\left(p : \frac{1}{n}\right)$$
$$= \{p\} \cup \left\{(x',y') \in X \mid y' < n - \sqrt{n^2 - (x'-x)^2},\ |x' - p| < \frac{1}{n}\right\},$$

then we call the space X thus defined *Ceder space.*

Definition 1.2. ([8]) Let X be a space and F a closed subset of X. An open cover $\mathcal{U}$ of $X \setminus F$ is called a *uniformly approaching anti-cover* of F in X if for each open subset O of X,

$$\overline{S(X \setminus O, \mathcal{U})} \cap F \cap O = \emptyset.$$

It is easy to see that an open cover $\mathcal{U}$ is a uniformly approaching anti-cover of F in X if and only if for each open set O of X there exists an open set O' of X such that

$$O' \cap F = O \cap F, \quad S(O', \mathcal{U}) \subset O.$$

or, there exists an subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that

$$V = (O \cap F) \cup \left(\bigcup \mathcal{U}_0\right)$$

is an open neighborhood of $O \cap F$ in X such that $V \subset O$.

And also it is easy to see that $\mathcal{U}$ is a uniformly approaching anti-cover of F in X if and only if for each point $p \in F$ and each basic neighborhood O of p in X, the following holds true:

$$\overline{S(X \setminus O, \mathcal{U})} \cap F \cap O = \emptyset.$$

We recall the Nagami's discussion [8] that every closed subset has a uniformly approaching anti-cover, which is used here a few times, referring to "Fundamental Method":

Fundamental Method ([8]): Let (X, d) be a metric space and F be a closed subset of X. For each $p \notin F$, let

$$U(p) = B\left(p, \frac{r(p)}{3}\right),$$

$r(p) = d(p, F)$ and $B(p, \varepsilon)$ is a open ball with center and radius ε. Then $\mathcal{U} = \{U(p) \mid p \in X \setminus F\}$ is an open cover of $X \setminus F$. To see that $\mathcal{U}$ is uniformly approaching, let O be an open set of X. Let

$$V = \{p \in O \mid d(p, X \setminus O) < d(p, X \setminus O)\}.$$

Then V is an open set of X such that $V \cap F = O \cap F \subset V$ and

$$S(X \setminus O, \mathcal{U}) \cap V = \emptyset.$$

Proposition 1.1. *In McAuley space X, X_0 has a uniformly approaching anti-cover.*

Proof. Let τ_{met} be the topology generated by the usual metric metric d on the Euclidean plane $\mathbb{R}^2$. Then τ_{met} is the subtopology of the topology τ of X and the following are easy to see:

1. X_0 is a closed subset of (X, τ_{met});
2. $\tau_{met}|X_0 = \tau|X_0$ and $\tau_{met}|(X \setminus X_0) = \tau|(X \setminus X_0)$.

Since (X, τ_{met}) is a metric space, X_0 has a uniformly approaching anti-cover $\mathcal{U}$ in (X, τ_{met}), constructed by the Fundamental Method stated above. We show that $\mathcal{U}$ is also a uniformly approaching anti-cover of X_0 in (X, τ). To this end, it suffices to show that for each $M(p : 1/k)$,there exists $m \in \mathbb{N}$ such that $k < m$ and

$$M\left(p : \frac{1}{m}\right) \cap S\left(X \setminus M\left(p : \frac{1}{k}\right), \mathcal{U}\right) = \emptyset.$$

Without loss of generality, we can assume $p = (0, 0)$. Obviously,

$$G = \left\{(x, y) \in X \mid y < \frac{1}{k}|x|\right\} = M\left(p : \frac{1}{k}\right) \setminus \{p\}$$

is open in (X, τ_{met}). Let

$$W = \left\{(x, y) \in X \mid y < \frac{1}{k + \sqrt{k^2 + 1}}|x|,\ y + |x| < \frac{1}{k}\right\}$$

Then W is an open subset of X such that

$$W = \{p \in G \mid d(p, X_0) < d(p, X \setminus G\}.$$

By the Fundamental Method, we have

$$S(X \setminus G, \mathcal{U}) \cap W = \emptyset.$$

Take $m \in \mathbb{N}$ such that $m > k + \sqrt{k^2 + 1}$. Then $M(p : 1/m)$ is an open neighborhood of p in X such that

$$M\left(X \setminus M\left(p : \frac{1}{k}\right), \mathcal{U}\right) \cap M\left(p : \frac{1}{m}\right) = \emptyset.$$

Hence $\mathcal{U}$ is a uniformly approaching anti-cover of X_0 in X. $\square$

Corollary 1.1. In Ceder space X, X_0 has a uniformly approaching anti-cover.

Proof. The discussion is almost the same to the case of McAuley space. Let $\mathcal{U}$ be the same family as in the above proof. Then we show that $\mathcal{U}$ is a uniformly approaching anti-cover of X_0 in this space X. To see it, it suffices to show that for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$C\left(p : \frac{1}{m}\right) \subset W \cup \{p\},$$

where

$$W = \left\{p \in X \mid d(p, X_0) < d\left(p, X \setminus C\left(p : \frac{1}{n}\right)\right)\right\}$$

Without loss of generality, we can show the case when $n = 1$ and $p = (0,0)$. By simple calculation, W coincides with

$$\{(x,y) \in X \mid 4y < x^2, \quad y + |x| < 1\}.$$

Then for a sufficiently large $m \in \mathbb{N}$, the following hold true:

$$C\left(p : \frac{1}{m}\right) \subset W \cup \{p\}.$$

By the Fundamental Method, we have

$$S\left(X \setminus C\left(p : \frac{1}{n}\right), \mathcal{V}\right) \cap C\left(p : \frac{1}{m}\right) = \emptyset.$$

This completes the proof. □

The following proof gives an answer to the problem proposed by Watson:

Problem 1.2.1 ([12, Problem 3.3.18]). Find a simple proof to the next theorem.

An essential point in the next proof is that if $\mathcal{U}$ is a locally finite cover of a space X, then $\{\bigcup \mathcal{U}_0 \mid \mathcal{U}_0 \subset \mathcal{U}\}$ is closure-preserving in X.

Theorem 1.1 (Nagata[10], Ceder[1]). *McAuley space and Ceder space are M_1-spaces.*

Proof. Since X_1 is a metrizable open subspace, there exists a σ-closure-preserving (in X) family $\mathcal{U}_0$ of open subsets of X forming a neighborhood base at each point of X_1 in X. By Proposition 1.1, there exists a uniformly approaching anti-cover $\mathcal{V}$ of X_0 in X. Without loss of generality, we can assume that $\mathcal{V}$ is locally finite in X_1. Let $(a,b) \in \mathbb{Q}^2$, $a < b$ be arbitrary. Let $\Delta(a,b)$ be the totality of subfamilies $\mathcal{V}(\delta)$ of $\mathcal{V}$ such that

$$U(\delta) = ((a,b) \times \{0\}) \cup \left(\bigcup \mathcal{V}(\delta)\right)$$

is an open neighborhood of $(a,b) \times \{0\}$ in X such that

$$\overline{U(\delta)} \subset \pi^{-1}[a,b],$$

where $\pi : X \to \mathbb{R}$ be the projection. Then

$$\mathcal{V}(a,b) = \{U(\delta) \mid \mathcal{V}(\delta) \in \Delta(a,b)\}$$

is a closure-preserving family of open subsets of X. It is easy to see that

$$\mathcal{W} = \bigcup\{\mathcal{V}(a,b) \mid (a,b) \in \mathbb{Q}^2,\ a < b\}$$

is a σ-closure-preserving family of open subsets of X forming a neighborhood base at each point of X_0 in X. Thus we have a σ-closure-preserving base

$$\mathcal{U} \cup \mathcal{W}$$

for X, proving that X is an M_1-space. □

1.3 The position of McAuley and Ceder space

Definition 1.3 ([8]). A space X is called a *D-space* if X is a paracompact σ-space such that each closed subset has a uniformly approaching anti-cover in X.

The following implications well known:

Metric space $\Longrightarrow$ Lašnev space $\Longrightarrow$ D-space-space in the sense of Nagami [8] $\Longrightarrow$ free L-space in the sense of Nagami [9] $\Longrightarrow$ M_3-μ-space $\Longrightarrow$ perfect image of an M_0-space $\Longrightarrow$ M_1-space

We give the position of both McAuley space and Ceder space in the sequence:

Theorem 1.2. *McAuley space and Ceder space are non-Lašnev D-spaces.*

Proof. Let X be a McAuley space. Then obviously X is a paracompact σ-space. Let F be a closed subset of X and let

$$F_0 = X_0 \cap F, \quad F_1 = X_1 \cap F.$$

By Proposition 1.1, there exists a uniformly approaching anti-cover $\mathcal{V}$ of X_0 in X. Without loss of generality, we can assume that $\mathcal{V}$ is locally finite in X_1. Since F is closed subset of X, for each point $p = (x,0) \in X_0 \setminus F_0$, there exists $k(x) \in \mathbb{N}$ such that

$$M\left(p : \frac{1}{k(x)}\right) \cap F = \emptyset.$$

Take $m(x) \in \mathbb{N}$ such that $m(x) > 3k(x)$ and

$$\begin{aligned} M\left(p : \frac{1}{m(x)}\right) \cap [&\mathbb{C}((x + 1/k(x), k(x)), k(x)) \\ &\cup \mathbb{C}((x - 1/k(x)), k(x)), k(x))] = \emptyset, \end{aligned} \tag{1.1}$$

where $\mathbb{C}(q,r)$'s are closed balls with center q and radius r. Let

$$U(p) = M\left(p : \frac{1}{m(x)}\right), \quad p = (x,0) \in X_0 \setminus F_0.$$

Then the family

$$\{U(p) \mid p = (x,0) \in X_0 \setminus F_0\}$$

has the following properties:

Claim 1: $\{U(p) \cap X_0 \mid p = (x, 0) \in X_0 \setminus F_0\}$ is a uniformly approaching anti-cover of F_0 in X_0.

For, since $m(x) > 3k(x)$, we have

$$\frac{1}{m(x)} < \frac{1}{3k(x)}, \quad (x, 0) \in X_0 \setminus F_0;$$

then we can use the Fundamental Method to show the next claim.

Claim 2:

$$U = \bigcup \{U(p) \mid p = (x, 0) \in X_0 \setminus F_0\} \tag{1.2}$$

is an open set of X such that

$$U \cap X_0 = X_0 \setminus F_0 \subset U \subset X \setminus F$$

and if O is an open set of X with $O \cap F_0 \neq \emptyset$, then

$$\overline{\pi(U \setminus O)} \cap (F_0 \cap O) = \emptyset,$$

where $\pi : X \to X_0$ is the projection.

Assume the contrary; then there exists

$$(x_0, 0) \in \overline{\pi(U \setminus O)} \cap (F_0 \cap O).$$

Since X_0 is a Fréchet space, there exists a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ of points of $U \setminus O$ such that $\lim_{n \to \infty} x_n = x_0$. Recalling the definition of U in (1.2), for each $n \in \mathbb{N}$ there exists $(x'_n, 0) \in X_0 \setminus F_0$ such that

$$(x_n, y_n) \in U((x'_n, 0)).$$

Since O is an open neighborhood of $(x_0, 0)$ in X, there exists $l \in \mathbb{N}$ such that

$$M\left((x_0, 0) : \frac{1}{l}\right) \subset O.$$

By the property (1.1), we can take $m_0 \in \mathbb{N}$ such that $m \geq m_0$ implies

$$M\left((x'_m, 0), \frac{1}{m(x'_m)}\right) \subset M\left((x_0, 0), \frac{1}{l}\right),$$

which implies $(x_n, y_n) \in O$. This is a contradiction to the fact $(x_n, y_n) \in U \setminus O$. This proves the claim.

Since $\mathcal{V}$ is uniformly approaching anti-cover of X_0 in X, there exists an open neighborhood U' of $X_0 \setminus F_0$ in X such that

$$\overline{S(U', \mathcal{V})} \cap X_1 \subset U.$$

Also, by this property, for each $p = (x, 0) \in X_0 \setminus F_0$, there exists $\mathcal{V}(x) \subset \mathcal{V}$ such that

$$V(p) = \left(x - \frac{1}{m(x)}, x + \frac{1}{m(x)}\right) \times \{0\} \cup \left(\bigcup \mathcal{V}(x)\right)$$

is an open neighborhood of $(x - 1/m(x), x + 1/m(x)) \times \{0\}$ in X and

$$V(p) \subset U' \cap U(p). \tag{1.3}$$

Note that

$$V(p) \subset \pi^{-1}\left(\left(x - \frac{1}{m(x)}, x + \frac{1}{m(x)}\right) \times \{0\}\right) \tag{1.4}$$

and that $\{V(p) \mid p = (x, 0) \in X_0 \setminus F_0\}$ is an open cover of $X_0 \setminus F_0$.

Since X_1 is a metric space, there exists a uniformly approaching anti-cover $\mathcal{W}$ of F_1 in X_1, which is locally finite in $X_1 \setminus F_1$. For each $p \in X_1 \setminus F_1$, let

$$W(p) = C(p, \mathcal{V}) \cap C(p, \mathcal{W}).$$

Then $W(p)$ is an open neighborhood of p in X such that

$$W(p) \cap F = \emptyset, \quad W(p) \cap X_0 = \emptyset.$$

Thus, finally we can construct an open cover

$$\mathcal{G} = \{V(p) \mid p \in X_0 \setminus F_0\} \cup \{W(p) \mid p \in X_1 \setminus F_1\}$$

of $X \setminus F$ in X. We show the next claim:

Claim 3: $\mathcal{G}$ is a uniformly approaching anti-cover of F in X.

To show it, let O be an open set of X. We consider the following two cases:

Case 1: $O \cap F_0 = \emptyset$.

Since $\mathcal{W}$ is a uniformly approaching anti-cover of F_1 in X, there exists an open set O' of X such that

$$O \cap F \subset O', \quad S(O', \mathcal{W}) \subset O \setminus \overline{S(U', \mathcal{V})}$$

By (1.3), we easily see

$$S(X \setminus O, \mathcal{G}) \cap O' = \emptyset.$$

Case 2: $O \cap F_0 \neq \emptyset$.

Since $\mathcal{W}$ is a uniformly approaching anti-cover of F_1 in X, there exists an open set Q of X such that

$$Q \cap F = O \cap F_1 \subset Q \subset S(Q, \mathcal{W}) \subset O \setminus \overline{S(U', \mathcal{V})}. \tag{1.5}$$

By Claim 2 and next by Claim 1, we can find an open set O_0 of X_0 satisfying the following:

$$\begin{aligned} &F_0 \subset O_0 \subset O \cap X_0, \ \overline{\pi(U' \setminus O)} \cap O_0 = \emptyset, \\ &S(O_0, \{V(p) \cap X_0 \mid p \in X_0 \setminus F_0\}) \subset (O \cap X_0) \setminus \overline{\pi(U' \setminus O)}. \end{aligned} \tag{1.6}$$

Since $\mathcal{V}$ is uniformly approaching, there exists an open set P of X such that

$$P \cap X_0 = O_0 \subset P, \ S(P, \mathcal{V}) \subset O, \ P \subset \pi^{-1}(O_0). \tag{1.7}$$

Then by (1.4) it is checked that

$$V(p) \cap P \neq \emptyset \iff (V(p) \cap X_0) \cap O_0 \neq \emptyset \tag{1.8}$$

for each $p = (x, 0) \in X_0 \setminus F_0$.

Finally we show that $S(P \cup Q, \mathcal{G}) \subset O$. Let $V(p)$, $p \in X_0 \setminus F_0$, be arbitrarily chosen from $\mathcal{G}$ and suppose $V(p) \cap (P \cup Q) \neq \emptyset$. By (1.3) and (1.5), we have $V(p) \cap Q = \emptyset$. Therefore $V(p) \cap P \neq \emptyset$. By (1.8),

$$(V(p) \cap X_0) \cap O_0 \neq \emptyset,$$

which combined with (1.6) implies $V(p) \cap (U' \setminus O) = \emptyset$. Therefore by (1.3) we have $V(p) \subset O$.

On the other hand, let $W(p)$, $p \in X_1 \setminus F_1$, be arbitrarily chosen from $\mathcal{G}$, and suppose $W(p) \cap (P \cup Q) \neq \emptyset$. If $W(p) \cap P \neq \emptyset$, then by (1.7) we have $W(p) \subset O$, and if $W(p) \cap Q \neq \emptyset$, then by (1.5), we have $W(p) \subset O$. In either case, we have the required inclusion. Hence X is a D-space.

The case of Ceder space is the same as above.

That both spaces are not metrizable is easily obtained from the Second Category Property of $\mathbb{R}$, and is well known. Since both spaces are first countable, they are not Lašnev spaces. This completes the proof. □

We cannot extend this result to general spaces which is the union $X = X_0 \cup X_1$ of a closed metrizable subset X_0 and metrizable subspace X_1 such that X_0 has a uniformly approaching anti-cover.

Example 1.3.1. Let $X = X_0 \cup X_1$ be a paracompact σ-space, where X_0 is a closed metrizable subset and X_1 is a metrizable subset of X such that X_0 has a uniformly approaching anti-cover. But X need not to be a D-space.

Proof. For each $n \in \mathbb{N}$, let $X(n)$ be a copy of the half plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

with the usual topology, and write it as

$$X(n) = \{((x, y), n) \mid (x, y) \in H\}.$$

Let X be the quotient space obtained from the topological sum $\oplus\{X(n) \mid n \in \mathbb{N}\}$ by identifying all $((x, 0), n)$, $n \in \mathbb{N}$ with a single point $< x, 0 >$. Let $f : \oplus\{X(n) \mid n \in \mathbb{N}\} \to X$ be the natural mapping. Let

$$X_0 = \{< x, 0 > \mid x \in \mathbb{R}\}, X_1 = X \setminus X_0.$$

Then obviously X_0 is a closed metrizable subset and X_1 is metrizable. Since X_0 has a uniformly approaching anti-cover in a metric space $f(X(n))$, X_0 has a uniformly approaching anti-cover in X.

We show that X is not a D-space. To this end, we show that $\{< x, 0 >\}$ does not have a uniformly approaching anti-cover. Let $\mathcal{U}$ be any open cover of $X \setminus \{< x, 0 >\}$. Let

$n \in \mathbb{N}$. There exists $U(n) \in \mathcal{U}$ such that $< 1/n, 0 > \in U(n)$. Since $U(n)$ is open in X, there exists $\varepsilon_n > 0$ such that

$$W(n) = f(\{((x,y),n) \in X(n) \mid y < \varepsilon_n\})$$

is an open neighborhood of X_0 in $f(X(n))$ such that $U(n) \setminus W(n) \neq \emptyset$. Then

$$W = f(\bigcup\{W(n) \mid n \in \mathbb{N}\})$$

is an open neighborhood of X_0 in X such that $U(n) \setminus W \neq \emptyset$ for each $n \in \mathbb{N}$. This implies that

$$< 0, 0 > \in \overline{S(X \setminus W, \mathcal{U})}.$$

Hence $\mathcal{U}$ is not a uniformly approaching anti-cover of $\{< 0, 0 >\}$ in X. $\square$

Chapter 2

Resolutions and D-spaces due to Nagami

2.1 Introduction

In this chapter, all spaces are assumed to be regular T_1 and all mappings to be continuous. For a space X, we represent the topology of X by $\tau(X)$. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural numbers, rational numbers and real numbers, respectively. We use "CP" to abbreviate the term "closure-preserving".

In this chapter, we study properties of McAuley space and Ceder space, defined later. The main result is that they are non-Lašnev D-spaces in the sense of Nagami [8]. Consequently, they lie between Lašnev spaces and L-spaces in the sense of Nagami. We also show that the subspace PN^{mca} of the resolution PN^{rb} of the Euclidean half plane has a σ-CP open base under the condition that the mappings $f_{(x,0)}$'s have suitable property. In fact, we can construct easily such an open base, which answers the problem proposed by Watson [12, Problem 3.3.18].

As for undefined terms, refer to [4].

2.2 The resolved plane PN^{rb}

First, we give the definition of resolutions of spaces following Watson [12, Def inion 3.1.32]. We can observe that McAuley space and Ceder space are considered to be the subspaces of PN^{rb}.

Definition 2.1. Suppose that X is a space and that $\{Y_x \mid x \in X\}$ are spaces and for each $x \in X$, $f_x : X \setminus \{x\} \to Y_x$ is a mapping. We call the space

$$Z = \bigcup\{\{x\} \times Y_x \mid x \in X\}$$

the resolution of X at each point $x \in X$ into Y_x by f_x if the topology of Z is given by the base

$$\{U \otimes_x V_x \mid U \in \tau(X),\ V_x \in \tau(Y_x),\ x \in X\},$$

where

$$U \otimes_x V_x = (\{x\} \times V_x) \cup \bigcup[\{x'\} \times Y_{x'} \mid x' \in U \cap f_x^{-1}(V_x)].$$

If $A \subset X$ and $Y_x = \{x^*\}$ is a singleton for each $x \in X \setminus A$ and $f_x : X \setminus \{x\} \to Y_x$ is the constant mapping, then we say that Z is a resolution of X at each point x of A by a mapping f_x into Y_x, and we denote the resultant resolution by

$$Z = R(X, f_x, Y_x : A).$$

For any $x \notin A$, the point $(x, x^*) \in Z$ is identified with $x \in X$.

Definition 2.2. We let the resolution

$$PN^{rb} = R(\mathbb{R} \times [0, \infty), f_{(x,0)}, Y_{(x,0)} \ : \ \mathbb{R} \times \{0\}),$$

where $\{Y_{(x,0)} : x \in \mathbb{R}\}$ are disjoint copies of $[0, 1]$ and for each $x \in \mathbb{R}$

$$f_{(x,0)} : X \setminus \{(x, 0)\} \to Y_{(x,0)}$$

satisfies

$$f_{(x,0)}(x', y') = \begin{cases} 0 & (y' = 0) \\ 1 & (x' = x) \end{cases}.$$

We say that PN^{rb} is the "resolved plane".

PN^{rb} is not metrizable. In fact, it has no G_δ-diagonal.

Theorem 2.1. *PN^{rb} has no G_δ-diagonal.*

Proof. Assume that PN^{rb} has a G_δ-diagonal sequence $(\mathcal{U}_n)_n$. For each $x \in \mathbb{R}$ since

$$p(x) = (x, 0, 0), \quad q(x) = (x, 0, 1)$$

are distinct, there exists $n(x) \in \mathbb{N}$ such that

$$q(x) \notin S(p(x), \mathcal{U}_{n(x)}).$$

By the Second Category Theorem, there exists $n_0 \in \mathbb{N}$ such that

$$\mathrm{Int_E}\overline{\{\mathrm{x} \in \mathbb{R} \mid \mathrm{n(x)} = \mathrm{n_0}\}}^{\mathrm{E}} \neq \emptyset,$$

where "E" means the topology in the usual sense. Take x_0 from the left term of the above. There exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points with $n(x_n) = n_0$ such that $\lim_{n \to \infty} x_n = x_0$. Since $\mathcal{U}_{n_0}$ is an open cover of PN^{rb}, there exists $U \in \mathcal{U}_{n_0}$ such that $(x_0, 0, 0) \in U$. By the definition of the topology,

$$(x_0, 0, 0) \in U^* \otimes_{(x_0,0)} V_{(x_0,0)} \subset U,$$

where U^*, $V_{(x_0,0)}$ are open neighborhoods of $(x_0, 0)$, 0 in $\mathbb{R} \times [0, \infty)$, $Y_{(x_0,0)}$, respectively, which implies

$$p(x_n), \ q(x_n) \in U^* \otimes_{(x_0,0)} V_{(x_0,0)}$$

for sufficiently large n. This is a contradiction. □

We give the definition of D-spaces in the sense of Nagami[8]:

Definition 2.3. [8] A space X is a "D-space" if X is a paracompact σ-space such that every closed subset F of X has a uniformly approaching anti-cover $\mathcal{U}$, that is, $\mathcal{U}$ is an open cover of $X \setminus F$ and for each $O \in \tau(X)$,

$$\overline{S(X \setminus O, \mathcal{U})} \cap (O \cap F) = \emptyset.$$

The position of D-spaces is given by the diagram:

Metric space⟶ Lašnev space⟶ D-space⟶ L-space⟶ free L-space
⟶ M_3-μ-space⟶ perfect image of an M_0-space⟶ M_1-space

That a metric space (X, d) is a D-space is shown as follows in [8]:
Let F be a closed subset of X. For each $x \in X \setminus F$, let $B(x, \varepsilon)$ be the open ball centered at x with radius ε. For an open cover $\mathcal{U}$, we define

$$\mathcal{U} = \left\{ B\left(p, \frac{1}{3} r(p)\right) \mid p \in X \setminus F \right\},$$

where $r(p) = d(p, F)$. Then $\mathcal{U}$ is an open cover of $X \setminus F$ and for each $O \in \tau(X)$,

$$S(X \setminus O, \mathcal{U}) \cap W = \emptyset,$$

where W is an open neighborhood of $O \cap F$ in X and

$$W = \{x \in O \mid d(x, F) < d(x, X \setminus O)\}.$$

Thus $\mathcal{U}$ is a uniformly approaching anti-cover of F in X. (We call the discussion "1/3-exercise".)

Definition 2.4. We call the subspace

$$(\mathbb{R} \times (0, \infty)) \cup ((\mathbb{R} \times \{0\}) \times \{0\}) \subset \mathrm{PN}^{\mathrm{rb}}$$

McAuley Plane and denote by PN^{mca}.

PN^{mca} is homeomorphic with the half plane

$$X = X_0 \cup X_1, \ X_0 = \mathbb{R} \times \{0\}, \ X_1 = \mathbb{R} \times (0, \infty)$$

topologized by defining the neighborhood bases as follows:
(i) Each point of X_1 has an open neighborhood base in the usual sense, and (ii) each point $p = (x, 0) \in X_0$ has an open neighborhood base

$$\{M(p, \varepsilon, \delta) \mid 0 < \varepsilon < 1, \ 0 < \delta < 1\},$$

where

$$M(p, \varepsilon, \delta) = \{p\} \cup \{(x', y') \in X \mid |x' - x| < \varepsilon, \ f_{(x,0)}(x', y') \in [0, \delta)\}.$$

Under suitable choices of $f_{(x,0)}$, PN^{mca} become McAuley space (Butterfly space) and Ceder space in [1], stated just below. Indeed, if we choose $f_{(x,0)}$, $x \in \mathbb{R}$ such that

$$f_{(x,0)}(x', y') = \sin \theta,$$

where θ is the angle from x-axis to (x', y'), then the resultant resolution is homeomorphic with McAuley space.

A similar observation can apply to the case for Ceder space. This space was introduced by Ceder [1].

"McAuley space" is topologized by defining as follows: (i) above and (ii)' each point $p = (x, 0) \in X_0$ has an open neighborhood base $\{M(p, \varepsilon) \mid 0 < \varepsilon < 1\}$, where

$$M(p, \varepsilon) = \{p\} \cup \{(x', y') \in X \mid y' < \varepsilon|x' - x| < \varepsilon^2\}.$$

"Ceder space" is topologized by defining as follows [1]: (i) above and (ii)" each point $p = (x, 0) \in X_0$ has an open neighborhood base $\{C(p, 1/n) \mid n \in \mathbb{N}\}$, where

$$C\left(p, \frac{1}{n}\right) = \{p\} \cup \left\{(x', y') \in X \mid y' < n - \sqrt{n^2 - (x' - x)^2},\ |x' - x| < \frac{1}{n}\right\}.$$

We define property $(*)$ to PN^{mca} as follows:

Definition 2.5. We say that PN^{mca} has property $(*)$ if the family $\mathcal{U} = \{B(p : (1/3)r(p)) \mid p \in X_1\}$, where $r(p) = d(p, X_0)$, $p \in X_1$, satisfies the following condition $(*)$:
$(*)$ For any ε with $0 < \varepsilon < 1$, there exists $\varepsilon' > 0$ such that

$$S\left(f^{-1}_{(x,0)}[0, \varepsilon'), \mathcal{U}\right) \subset f^{-1}_{(x,0)}[0, \varepsilon).$$

By the 1/3-exercise, it is obvious that McAuley space satisfies property $(*)$. We also check easily that Ceder space satisfies property $(*)$.

Theorem 2.2. *If* PN^{mca} *has the property* $(*)$, then it has a σ-CP base.

Proof. Since $X = \mathrm{PN}^{mca}$ is a paracompact σ-space, there exists a locally finite open (in X) refinement $\mathcal{V}$ of $\mathcal{U}$. For each pair $(a, b) \in \mathbb{Q}^2$, $a < b$, let $\{\mathcal{V}_\alpha \mid \alpha \in \Delta(a, b)\}$ be the totality of subfamilies of $\mathcal{V}$ such that

$$W(a, b, \alpha) = ((a, b) \times \{0\}) \cup (\bigcup \mathcal{V}_\alpha)$$

is an open neighborhood of $(a, b) \times \{0\}$ on X satisfying

$$W(a, b, \alpha) \subset (a, b) \times [0, \infty).$$

Then

$$\mathcal{W}(a, b) = \{W(a, b, \alpha) \mid \alpha \in \Delta(a, b)\}$$

is a CP family of open subsets of X, because $\mathcal{V}$ is locally finite in X. We show that

$$\mathcal{W}_0 = \bigcup\{\mathcal{W}(a, b) \mid a,\ b \in \mathbb{Q}, a < b\}$$

is a local base at each point of X_0 in X. Let $p = (x, 0) \in X_0$ and O an open neighborhood of p in X. Then there exist $\varepsilon > 0$, $a, b \in \mathbb{Q}$ with $a < b$ such that

$$p \in (a, b) \times [0, \infty) \otimes_{(x,0)} [0, \varepsilon) \subset O.$$

By property $(*)$, there exists $\varepsilon' > 0$ such that

$$S(f^{-1}_{(x,0)}([0,\varepsilon')),\mathcal{U}) \subset f^{-1}_{(x,0)}([0,\varepsilon)).$$

By the 1/3-exercise, there exists $\mathcal{V}_\alpha$, $\alpha \in \Delta(a,b)$ such that

$$p \in W(a,b,\alpha) \subset (a,b) \times [0,\infty) \otimes_{(x,0)} [0,\varepsilon).$$

Since X_1 is an open metric subspace, there exists a σ-locally finite (in X) family of open subsets, forming a local base at each point of X, in X. Thus, we have constructed a σ-CP base $\mathcal{W}_0 \cup \mathcal{W}_1$ for X. □

Remark 2.2.1. Since both McAuley space and Ceder space have the property $(*)$, the proof above gives the method of construction of σ-CP open base. The result itself is already known by Ceder [1] for Ceder space and by Nagata [10] for McAuley space. But the proof of [1] is too lengthy and the proof of [10] is not given. So, it is natural that the following problem is proposed by Watson.

Problem [12, problem 3.3.18]: Find a simple proof to their result that PN^{mca} has a σ-CP base when a suitable choice of $f_{(x,0)}$'s is used in the construction of PN^{rb}.

Hence, this gives an answer to the problem proposed by Watson [12, Problem 3.3.18], in a more general form.

Remark 2.2.2. Since $\mathcal{V}$ is locally finite in X_1, we can easily observe that for the family $\mathcal{W}(a,b)$, there exists a σ-discrete family $\mathcal{H}$ of closed subsets of X such that $\mathcal{W}(a,b)$ is $\mathcal{H}$-preserving in both sides in the sense of Mizokami [5]. Hence PN^{mca} with property $(*)$ is an M_3-μ-space.

2.3 The main result

Lemma 2.3.1. *Let $p_0 = (x_0, 0)$ and $k \in \mathbb{N}$ be fixed. For each point $p = (x,0)$, $x \neq x_0$, let*

$$a(x) = \frac{1}{e^{\frac{1}{|x-x_0|}}}.$$

Then there exists $x' \in \mathbb{R}$ with $0 < x'$, $|x - x'| < 1/k$ such that $|x - x_0| < |x' - x_0|$ implies

$$M((x,0), a(x)) \subset M\left(p_0, \frac{1}{k}\right)$$

Proof. Without loss of generality, we can assume $(x_0, 0) = (0,0)$. Since the heights of two right triangles forming $M((x,0),a(x))$ are $a(x)^2 = 1/(e^{1/x})^2$, it suffices to show that for sufficiently small $x > 0$, the following holds:

$$\frac{1}{k}\left(x - \frac{1}{e^{\frac{1}{x}}}\right) > \left(\frac{1}{e^{\frac{1}{x}}}\right)^2,$$

that is,

$$P(x) = e^{\frac{1}{x}}\left(xe^{\frac{1}{x}} - 1\right) - k$$

is positive. To see it, we calculate $P(x)$ as follows: By Maclaurin expansion, we have

$$\begin{aligned} P(x) &= xe^{\frac{2}{x}} - e^{\frac{1}{x}} - k \\ &= x\left(1 + \frac{2}{x} + \frac{2}{x^2} + \sum_{n\geq 3} \frac{1}{n!}\left(\frac{2}{x}\right)^n\right) - \left(1 + \frac{1}{x} + \sum_{n\geq 2} \frac{1}{n!}\frac{1}{x^n}\right) - k \\ &= x + 2 + \frac{2}{x} + \sum_{n\geq 2} \frac{2^{n+1}}{(n+1)!}\frac{1}{x^n} - \left(1 + \frac{1}{x} + \sum_{n\geq 2} \frac{1}{n!}\frac{1}{x^n}\right) - k \\ &= x + 1 + \frac{1}{x} + \sum_{n\geq 2} \frac{2^{n+1} - (n+1)}{(n+1)!x^n} - k, \end{aligned}$$

where $(2^{n+1} - (n+1))/(n+1)!x^n > 0$ for $n \geq 2$ and $0 < x < 1$, from which we have $\lim_{x\to o+o} P(x) = \infty$. Hence there exists $x' > 0$ such that $0 < x < x'$ implies that $P(x)$ is positive. □

Theorem 2.3. *McAuley space is a D-space in the sense of Nagami.*

Proof. Let H be a closed subset of McAuley space X. Then we show that H has a uniformly approaching anti-cover. Let $p = (x, y) \in X \setminus H$. If $p \in X_1$, then let

$$V(p) = B(p, \frac{1}{3}r(p)),$$

where $r(p) = d(p, H \cup X_0)$. If $p = (x, 0) \in X_0 \setminus H_0$, $H_0 = X_0 \cap H$, then there exists $a(p)$ with $0 < a(p) < 1$ such that

$$M(p, a(p)) \cap H = \emptyset.$$

Let

$$b(p) = \frac{1}{e^{\frac{1}{d(p,H_0)}}}.$$

Let

$$V(p) = M\left(p, \frac{1}{3}r(p)\right),$$

where $r(p) = \min\{a(p),\ b(p)\}$. Then

$$\mathcal{V} = \{V(p) \mid p \in X \setminus H\}$$

is an open cover of $X \setminus H$. To show that $\mathcal{V}$ is uniformly approaching, we settle the following claim:

Claim:

$$\overline{\bigcup\{V(p) \mid p \in X_0 \setminus H_0\}} \cap H_1 = \emptyset,$$

where $H_1 = H \cap X_1$.

Assume the contrary. Take a point $p_0 = (x_0, y_0)$ from the left term above. There exists a sequence $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to p_0$ as $n \to \infty$ and

$$p_n = (x_n, y_n) \in M\left(p'_n, \frac{1}{3}r(p'_n)\right), \quad n \in \mathbb{N}$$

Since $y_0 > 0$, $B(p_0, (1/3)y_0)$ contains (x_n, y_n) for large numbers n. From this, we can show that $(x_0, y_0) \in M(p'_n, r(p'_n))$, which is a contradiction. To show it, without loss of generality we can assume that $p'_n = (0,0)$ and $x_0 > 0$. We let $r(p'_n) = 3\varepsilon$. First, we show that $x_0 < 2\varepsilon$. Since

$$(x_n, y_n) \in B(p_0, \frac{1}{3}y_0) \cap M((0,0), \varepsilon),$$

we have

$$x_0 < \varepsilon + \frac{1}{3}y_0, \quad y_n < \varepsilon x_n < \varepsilon^2 \tag{2.1}$$

Since $d(p_0, X_0) \leq d(p_0, p_n) + d(p_n, X_0)$, we have $y_0 < (1/3)y_0 + y_n$, i.e.,

$$y_n > \frac{2}{3}y_0. \tag{2.2}$$

From (2.1) and (2.2), the following holds:

$$\begin{aligned} x_0 &< \varepsilon + \frac{1}{3}y_0 \quad \text{(by (2.1))} \\ &< \varepsilon + \frac{1}{2}y_n \quad \text{(by (2.2))} \\ &< \varepsilon + \frac{1}{2}\varepsilon^2 \quad \text{(by (2.1))} \\ &< \frac{3}{2}\varepsilon < 2\varepsilon \end{aligned}$$

By the triangle inequality,

$$d(p_0, X_0) \leq d(p_0, p_n) + d(p_n, X_0).$$

Since $p_n \in M((0,0), \varepsilon)$, $d(p_n, X_0) < \varepsilon(x_0 + \delta)$, where $\delta = x_n - x_0$. Therefore we have

$$y_0 < \frac{1}{3}y_0 + \varepsilon(x_0 + \delta) = \frac{1}{3}y_0 + \varepsilon x_0 + \varepsilon\delta,$$

and

$$|\delta| < \frac{1}{3}y_0, \quad 0 < \varepsilon < 1.$$

These imply

$$y_0 < \frac{1}{3}y_0 + \varepsilon x_0 + \frac{1}{3}y_0 = \frac{2}{3}y_0 + \varepsilon x_0.$$

Hence, we have $y_0 < 3\varepsilon x_0$.

Consequently, we have $(x_0, y_0) \in M(p, 3\varepsilon)$, which is a contradiction.

Now, we show that $\mathcal{V}$ is uniformly approaching. Let $p \in O \in \tau(X)$ and $p \in H$. Suppose the two cases:

As the first case, let $p = (x, y) \in X_1$. By the 1/3-exercise, we have

$$\overline{S(X \setminus O, \mathcal{V}_1)} \cap O \cap H = \emptyset,$$

where

$$\mathcal{V}_1 = \{V(p) \mid p \in X_1 \setminus H\}.$$

By the claim, $\overline{\bigcup \mathcal{V}_0} \cap H = \emptyset$, where

$$\mathcal{V}_0 = \mathcal{V} \setminus \mathcal{V}_1 = \{V(p) \mid p \in X_0 \setminus H_0\}.$$

Therefore we have

$$\overline{S(X \setminus O, \mathcal{V})} \cap O \cap V = \emptyset$$

for some open neighborhood V of p in X such that $V \subset O$.

We consider the remaining case that $p = (x, 0) \in H_0$. If $p \in \operatorname{Int} H_0$, then we can take $M(p, \varepsilon)$ such that

$$p \in M(p, \varepsilon) \subset O, \quad (x - \varepsilon, x + \varepsilon) \subset \operatorname{Int} H_0.$$

By the 1/3-exercise, we have

$$S(X \setminus O, \mathcal{V}_1) \cap M\left(p, \frac{1}{3}\varepsilon\right) = \emptyset$$

and $(x - \varepsilon, x + \varepsilon) \subset \operatorname{Int} H_0$ implies

$$M\left(p, \frac{\varepsilon}{3}\right) \cap \left(\bigcup \mathcal{V}_0\right) = \emptyset.$$

Hence we have

$$M\left(p, \frac{\varepsilon}{3}\right) \cap S(X \setminus O, \mathcal{V}_0) = \emptyset.$$

Suppose $p \notin \operatorname{Int} H_0$ and take $M(p, \varepsilon)$ such that

$$p \in M(p, \varepsilon) \subset O.$$

By the 1/3-exercise, we have

$$S(X \setminus O, \mathcal{V}_1) \cap M\left(p, \frac{\varepsilon}{3}\right) = \emptyset. \tag{2.3}$$

By Lemma 2.3.1, there exists $\varepsilon_0 > 0$ such that if

$$|x' - x| < \varepsilon_0, \ p' = (x', 0) \notin H,$$

then

$$M\left(p', \frac{1}{3} r(p')\right) \subset M(p, \varepsilon).$$

Let $\varepsilon' = \min\{\varepsilon, \varepsilon_0\}$. Then we have

$$S(X \setminus O, \mathcal{V}_0) \cap M\left(p, \frac{1}{3}\varepsilon'\right) = \emptyset \tag{2.4}$$

By (2.3) and (2.4), we can say that $\mathcal{V}$ is uniformly approaching. □

Remark 2.3.1. By the Second Category Theorem, it is routine to check that PN^{mca} does not have a σ-locally finite base. Since it is first countable, it is not a Lašnev space.

Remark 2.3.2. It is easy to see that the essential discussion of the above proof can apply to the case of Ceder space and the corresponding result to Lemma2.3.1 hold true for the case of Ceder spaces. Hence we can say that Ceder space is a non-Lašnev D-space.

Chapter 3

Perfect mappings between resolutions

3.1 Introduction

In this chapter, all spaces are assumed to be regular T_1 and all mappings to be continuous. $\mathbb{N}$ denotes always all natural numbers. For a space X, $\tau(X)$ denotes the topology of X. The concept of resolutions of spaces was originally given by Fedorčuk [3] and Watson [12] brought it in the limelight. Watson showed how nice properties of topological spaces can be destroyed by taking resolutions and that many counterexamples are obtained by taking resolutions.

We give the definition of the resolution of a space. Let X and Y_x, $x \in X$, be spaces and for each x, $f_x : X \setminus \{x\} \to Y_x$ a mapping. We endow the set

$$Z = \bigcup\{\{x\} \times Y_x \mid x \in X\}$$

with the topology whose base consists of

$$\{U \otimes_x V \mid U \in \tau(X),\ V \in \tau(Y_x),\ x \in X\},$$

where

$$U \otimes_x V = (\{x\} \times V) \cup \bigcup\{\{x'\} \times Y_{x'} \mid x' \in U \cap f_x^{-1}(V)\},$$

for $U \in \tau(X),\ V \in \tau(Y_x)$.

We call Z thus defined the *resolution* of a space X (at each point $x \in X$ into Y_x by the mapping f_x), and simply denote this by

$$Z = R(X, f_x, Y_x).$$

For simplicity, we frequently write $U \otimes V$ in place of $U \otimes_x V$ above, if no confusion. We have to point out that any resolution of regular spaces is also regular, [2].

In this chapter, we resolve a metrizable space X at each point $x \in X$ into a paracompact M-space Y_x in the sense of Morita, [7], by a mapping f_x, and show that under the condition stated below $R(X, f_x, Y_x)$ is also a paracompact M-space. For it, we show that perfect mappings $g_x : Y_x \to Z_x$, $x \in X$ induces the perfect mapping

$$\Phi : R(X, f_x, Y_x) \to R(X, g_x f_x, Z_x).$$

3.2 The resolutions of paracompact M-spaces

We call a subset Λ of a space X *F_σ-discrete* in X if $\Lambda = \bigcup\{\Lambda_n \mid n \in \mathbb{N}\}$, where each Λ_n is discrete and closed in X.

Let

$$\Lambda = \{x \in X \mid |Y_x| > 1\}$$

In general, it is well known that the condition that Λ is F_σ-discrete in X helps well as for the preservation of generalized metric properties of X, Y_x, $x \in X$ to the resolutions.

In fact, Richardson and Watson [11] showed that under the condition that Λ is F_σ-discrete in X, the resolution $R(X, f_x, Y_x)$ is metrizable if and only if X and all Y_x, $x \in X$ are metrizable. Mizokami and Suwada gave some similar results with the same direction in [6].

Suppose that we are give a space X, spaces Y_x, Z_x, $x \in X$ and mappings $f_x : X\backslash\{x\} \to Y_x$, $x \in X$. For each $x \in X$, let $g_x : Y_x \to Z_x$ be a mapping. We write the resolutions of X as follows:

$$S = R(X, f_x, Y_x), \quad T = R(X, g_x f_x, Z_x)$$

We define a mapping $\Phi : S \to T$ as follows:

$$\Phi(x, y) = (x, g_x(y)), \quad (x, y) \in S$$

Theorem 3.1. *If for each $x \in X$, $g_x : Y_x \to Z_x$ is a perfect mapping, then so is Φ.*

Proof. Let $\Phi : S \to T$ be a mapping defined above. First, we show that Φ is continuous. To this end, it suffices to show that for each $U \in \tau(X)$ and $V_x \in \tau(Z_x)$, $x \in X$, $\Phi^{-1}(U \otimes_x V_x)$ is open in S. Let (x', y') be an arbitrary point of $\Phi^{-1}(U \otimes_x V_x)$. Then we have

$$\Phi(x', y') = (x', g_x(y')) \in U \otimes_x V_x.$$

If $x = x'$, then $y' \in g_x^{-1}(V_x)$. If $x' \neq x$, then

$$x' \in U \cap f_x^{-1}(g_x^{-1}(V_x)).$$

These imply that $U \otimes_x g_x^{-1}(V_x)$ is an open neighborhood of (x', y') in S. Moreover, we can observe

$$\Phi(U \otimes_x g_x^{-1}(V_x)) \subset U \otimes_x V_x.$$

In fact, let $(x', y') \in U \otimes_x g_x^{-1}(V_x)$. If $x' = x$, then $y' \in g_x^{-1}(V_x)$, which implies

$$(x', g_x(y')) \in \{x\} \times V_x.$$

If $x' \neq x$, then

$$x' \in U \cap f_x^{-1}(g_x^{-1}(V_x)),$$

which implies

$$(x', g_x(y')) \in U \otimes_x V_x.$$

Hence Φ is continuous.

For each $(x, z) \in T$,

$$\Phi^{-1}(x, z) = \{x\} \times g_x^{-1}(z).$$

Since g_x is a perfect mapping, $\Phi^{-1}(x,z)$ is compact. Thus it remains to show that Φ is a closed mapping. Let O be an open neighborhood of $\Phi^{-1}(x,z)$ in S. Since $\Phi^{-1}(x,z)$ is compact, there exist $V_1,\ V_2,\ \cdots,\ V_k \in \tau(Y_x)$ and $U \in \tau(X)$ satisfying the following:

$$\Phi^{-1}(x,z) \subset \bigcup_{i=1}^{k}(U \otimes_x V_i) \subset O$$

This implies that $\{V_i \,|\, i = 1,\ \cdots,\ k\}$ is an open cover of $g_x^{-1}(z)$ in Y_x. Since $g_x : Y_x \to Z_x$ is closed, there exists an open neighborhood V of z in Z_x such that

$$g_x^{-1}(z) \subset g_x^{-1}(V) \subset \bigcup_{i=1}^{k} V_i \tag{1}$$

Then we show the following:

$$\Phi^{-1}(U \otimes_x V) \subset \bigcup_{i=1}^{k}(U \otimes_x V_i) \tag{2}$$

To see this, let (x', y') be an arbitrary point of $\Phi^{-1}(U \otimes_x V)$. Then we have

$$\Phi(x', y') = (x', g_x(y')) \in U \otimes_x V$$

If $x = x'$, then $g_x(y') \in V$, which combined with (1) implies $y' \in V_i$ for some i. Therefore we have $(x', y') \in U \otimes_x V_i$. If $x' \neq x$, then

$$x' \in U \cap (g_x f_x)^{-1}(V_x),$$

which combined with (1) implies

$$x' \in U \cap f_x^{-1}(V_i)$$

for some i. Hence we have

$$(x', y') \in U \otimes_x V_i,$$

proving the validity of (2). This completes the proof. □

M-spaces are introduced by Morita, [7]. The definition is as follows:

Definition 3.1. ([4, Definition 3.5]) A space X is called an M-*space* if there exists a sequence $\{\mathcal{U}_n \,|\, n \in \mathbb{N}\}$ satisfying the following two conditions:

(1) For each $n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$;

(2) if $x_n \in S(x, \mathcal{U}_n)$, $n \in \mathbb{N}$, then $\{x_n \,|\, n \in \mathbb{N}\}$ clusters in X.

A paracompact M-space is characterized to be a perfect pre-image of a metrizable space, [4, Corollary 3.7], and also to be a paracompact p-space in the sense of Arhangel'skii. In virtue of the above theorem, we resolve a metrizable space X into paracompact M-spaces as follows:

Theorem 3.2. *Let $S = R(X, f_x, Y_x)$ and let Λ be F_σ-discrete in X. If X is metrizable space and for each $x \in X$, Y_x is a paracompact M-space, then the resolution S is also a paracompact M-space.*

Proof. For each $x \in X$, there exists a perfect mapping g_x of Y_x onto a metrizable space Z_x. By the above theorem,

$$\Phi : S \to T = R(X, g_x f_x, Z_x)$$

is a perfect mapping. The condition on Λ implies that T is metrizable, [11]. Hence S is a paracompact M-space. □

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