Finite switchboard state machines and fuzzy finite switchboard state machines

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Preface

In the 20th century, computer sciences have been developing and influencing various fields of many sciences. In recent years, studies from a new view of social affairs have been demanded while the complication of human society has grown. In the meantime the theory of machines has been growing by men of abilities, for instance, Claude Elwood Shannon, Alan Mathison Turing, John von Neumann. The aim of so-called *system theory* was to build up ideas and tools of various systems such as electrical engineering, physiology, theoretical biology, sociology, linguistics.

The theory of information, the theory of control, the theory of finite state machines and the theory of fuzzy systems can be regarded as some of the essential parts of system theory. The areas of mathematics that is mostly used in these theories are probability theories and modern algebra. The mathematical tools that is of most use for this paper is modern algebra. For some hundred years, algebra has developed in a lot of directions. But the growing of theory of machines has provided human beings with new motivation for the development of algebra. Thus W.M.L.Holcombe [1] upon which the former part of this paper is based deals with discussions on the algebraic realm of automata theory.

Chapter 1 begins with some of elementary material concerning the theory of algebraic automata theory. After introducing the concept of state machine at first, semigroups of state machine are formed on sets of state. Thus transformation semigroups are associated with the state machines by the semigroups of state machines. State machine homomorphisms leads to quotients of state machine, and so to transformation homomorphisms. When the functional properties of the state machines are discussed, the concept of covering is more useful than that of homomorphism.

For example, transformation semigroups of A state machines are able to be covered by transformation semigroups of B state machines if A state machines are covered by B state machines. Product of state machines and of transformation semigroup can produce many interesting properties in relation to coverings. Some of covering theorems can be rewritten by the concept of homomorphism. Their examples are picked up to reveal the usefulness of the algebraic automata theory. Some kind of products that are dealt with in this paper are the most important when we discuss finite state machines and transformation semigroups. We define restricted direct products, full direct products, cascade products and wreath products with respect to finite state machines and transformation semigroups. As to wreath products, it is known that wreath products of two transformation semigroups are taransformation semigroups. And transformation semigroups of wreath product of two kinds of finite state machines are able to be covered by wreath products of two kinds of transformation semigroups of state machines. As to direct products, transformation semigroups of full direct product of two kinds of finite state machines are able to be covered by full direct products of two kinds of transformation semigroups of state machines. As to relations between full direct products and wreath products, transformation semigroups of full direct product of two kinds of finite state machines are able to be covered by wreath products of two kinds of transformation semigroups of state machines.

Chapter 2 examines some elementary properties of switchboard state machines and switchboard transformation semigroups. After defining switchboard state machines by binding the concept of switching state machines and commutative state machines together, switchboard transformation semigroups are defined the same.

Examples switchboard state machines, switching state machines and commutative state machines are picked up. It is revealed that transformation semigroups of switchboard state machines are switchboard transformation semigroups. By introducing switching relations on switchboard state machines, switching classes are defined and quotients of state machine led by the switching classes become switchboard state machines, too. We call the quotients switchboard state machines for small sigma in large sigma. Switchboard state machines for small sigma in large sigma can be covered by the switchboard state machines.

We define restricted cascade products between two complete state machines and present some examples of restricted cascade products. Through semigroup epimorphisms from one switchboard state machine to another switchboard state machines, it revealed that cascade products of the two switchboard state machines being switchboard state machines are equal to the two state machines being switchboard state machines. By counter examples, cascade products of transformation semigroups of two state machines cannot be covered by transformation semigroups of cascade products of two state machines. However on an special conditions, transformation semigroups of restricted cascade products of state machines and switchboard state machines are isomorphic to restricted cascade products of transformation semigroups of state machines and transformation semigroups of switchboard state machines. This theorem produces some corollaries with respect to the other products.

We study cartesian compositions of state machines. After presenting some examples of cartesian compositions of state machines, it is revealed that cartesian compositions of switchboard state machines are switchboard state machines, too. And we examine relations among cartesian products and the other products. All Lemmas and Propositions in Chapter 2 are due to [11].

Quite a long time after L.A.Zadeh introducing the concept of fuzzy sets, the field of fuzzy sciences was regarded as some exotic field of research. But the resent success with consumer products involving fuzzy tools causes a growing interest of not only mathematicians but also engineers.

Chapter 3 introduces the concept of fuzzy finite state machines and fuzzy transformation semigroups at first. After some examples of fuzzy finite state machines and fuzzy transformation semigroups, some concepts of homomorphism of fuzzy finite state machines and fuzzy transformation semigroups are introduced. Among them, strong homomorphisms are most important. For example, there exists an isomorphism from some quotients of A fuzzy finite state machines to B fuzzy finite state machines if there exists an onto strong homomorphism from A fuzzy finite state machines to B fuzzy finite state machines.

We introduce the concept of fuzzy finite state machines and fuzzy transformation semigroups. Those are some kinds of fuzzilization of finite state machines and transformation semigroups. The idea of switching homomorphism is introduced, and some properties which are analogous to those obtained by Malik, Moderson and Sen [10] are examined. All Lemmas and Propositions in Chapter 3 are due to [12].

Abstract

This paper deals with switchboard state machines which are specialized finite state machines, and deals with fuzzy finite switchboard state machines which are fuzzilizations of fuzzy finite state machines. The first half of this paper is based upon W.M.L.Holcombe [1], which deals with discussions on the algebraic realm of automata theory.

In regard to transformation semigroups, see the book [5].

After introducing the concept of state machine at first, semigroups of state machine are formed on sets of state. Thus transformation semigroups are associated with the state machines by the semigroups of state machines. State machine homomorphisms leads to quotients of state machine, and so to transformation homomorphisms.

When the functional properties of the state machines are discussed, the concept of covering is more useful than that of homomorphism. Products of state machines and of transformation semigroups can produce many interesting properties in relation to coverings.

The second half of this paper deals with restricted types of finite state machines. Switchboard state machines and switchboard transformation semigroup are introduced at first. Some types of products are characterized by these concepts. New type of products are contained in those. As to these new types of products, some properties in relation to state machines and transformation semigroups are discussed.

Lastly, the concept of fuzzy finite switchboard state machines and fuzzy transformation semigroups are introduced. The idea of switching homomorphism is introduced, and some properties which are analogous to those obtained by Malik, Mordeson and Sen [10] are examined.

Their examples are picked up to reveal the usefulness of the algebraic automata theory.

1. Finite state machines

1.1. Introduction

The theory of automata originated in some former theories. They contain the theory of Turing machine, computer and Kleen's theory. The theory of automata has developed in the last twenty years, and the mathematical theories has been growing along together. One of them is the algebraic automata theory. The theory of machines has been applied to computer systems, linguistics, biology, psychology, biochemistry, sociology, etc. Most of all,the theory using algebraic techniques has developed recently. In many systems environmental stimuli change organisms. We can deem many of interaction in those as discrete and finite while some of those behavier are continuous and infinite. Some kinds of simple systems are called state machines. These have strong relation to transformation semigroups. In regard to transformation semigroups, various important results have ever been held including The Covering Lemma paper [3].

To think systems simply, we identify the set Q with *internal states* and idealize environmental input to be *input alphabet* Σ. We define a partial function $F : Q \times \Sigma \to Q$ in such a way that $F((q, \sigma)) = q'$ where $q \in Q$, $\sigma \in \Sigma$ and q' is the result of applying σ to the system in state q.

A *state machine* is a triple $M = (Q, \Sigma, F)$ where Q and Σ are finite sets and F is a partial function $F : Q \times \Sigma \to Q$. We consider the set Σ^+ of all words in the alphabet Σ, define a relation $\sim$ on Σ^+ by

$$\alpha \sim \beta \Leftrightarrow F_\alpha = F_\beta$$

where $\alpha, \beta \in \Sigma'$.

$M = (Q, \Sigma, F)$ is called *complete* if the partial function $F : Q \times \Sigma \to Q$ is a function.

$\sim$ is a congruence relation, so we construct the quotient semigroup $\Sigma^+ / \sim$ and call it the *semigroup of the state machine* M, denoted $\mathbf{S}(M)$. The elements of $\mathbf{S}(M)$ are equivalence classes $[\alpha], \alpha \in \Sigma^+$. Λ is called the *null word* satisfying $a\Lambda = \Lambda a = a$ for $a \in \Sigma^+$. Define the *free monoid generated by the set* Σ by $\Sigma^* = \Sigma^+ \cup \{\Lambda\}$.

Almost all Definitions, Theorems and Propositions in this chapter are due to W.M.Holcombe [1].

Examples

(1) Some simple cases are where $|Q| = 2$ and $|\Sigma| = 1$. Let $M = (Q, \Sigma, F)$ be a complete state machine where $Q = \{0, 1\}$ and $\Sigma = \{\sigma\}$.

(a)

F	0	1
σ	0	1

(b)

F	0	1
σ	1	1

(c)

F	0	1
σ	0	0

(d)

F	0	1
σ	1	0

(2) A little comlicated cases are where $|Q| = 2$ and $|\Sigma| = 2$. Let $M = (Q, \Sigma, F)$ be a complete state machine where $Q = \{0, 1\}$ and $\Sigma = \{\sigma, \tau\}$.

(a)

F	0	1
σ	0	1
τ	1	1

(b)

F	0	1
σ	0	1
τ	0	0

(c)

F	0	1
σ	0	1
τ	1	0

(d)

F	0	1
σ	1	1
τ	0	0

(e)

F	0	1
σ	1	1
τ	1	0

(f)

F	0	1
σ	0	0
τ	1	0

(3) Another little comlicated cases are where $|Q| = 3$ and $|\Sigma| = 1$. Let $M = (Q, \Sigma, F)$ be a complete state machine where $Q = \{0, 1, 2\}$ and $\Sigma = \{\sigma\}$.

(a)

F	0	1	2
σ	0	1	2

(b)

F	0	1	2
σ	0	2	1

(c)

F	0	1	2
σ	0	0	1

(d)

F	0	1	2
σ	0	1	0

(e)

F	0	1	2
σ	0	1	1

And so forth. All kinds of state machines where $|Q| = 3$, $|\Sigma| = 1$ are 27.

1.2. Homomorphisms

Finite state machines have strong relation to transformation semigroups. Some properties are dealt with through the conception of homomorphism here. Admissible relations are necessary and sufficient conditions to construct quotient state machines and quotient transformation semigroups. There exists a sort of homomorphism theorem between state machines and their quotients, as well as between transformation semigroups and their quotients.

Proposition 1.2.1
Let $M = (Q, \Sigma, F)$ be a state mashine and $\langle \mathbf{F}(M) \rangle$ the subsemigroup of $\mathbf{PF}(Q)$ generated by $\{F_\sigma | \sigma \in \Sigma\}$, then

$$\langle \mathbf{F}(M) \rangle \cong \mathbf{S}(M) = \Sigma^+/\sim .$$

Furthermore $\mathbf{S}(M)$ is a finite semigroup.

Proof Define $\theta : \Sigma^+/\sim \to \langle \mathbf{F}(M) \rangle$ by $\theta([\alpha]) = F_\alpha$ for $\forall \alpha \in \Sigma^+$. Clearly θ is well-defined. Let $\gamma \in \Sigma^+$. As we can indicate $F = F_\gamma$ for $\forall F \in \langle \mathbf{F}(M) \rangle$, there exists $[\gamma] \in \Sigma^+/\sim$. Hence θ is surjective.
Let $\theta([\alpha]) = \theta([\beta])$. Then $F_\alpha = F_\beta \Leftrightarrow [\alpha] = [\beta]$. Hence θ is injective.
Let $[\alpha], [\beta] \in \Sigma^+/\sim$. Then

$$\begin{aligned} \theta([\alpha][\beta]) &= F_{\alpha\beta} \\ &= F_\alpha F_\beta \\ &= \theta([\alpha])\theta([\beta]). \end{aligned}$$

Hence θ is a homomorphism.
Since $|Q| < \infty$, clearly $|\mathbf{PF}(Q)| < \infty$. Hence $|\mathbf{S}(M)| \leq |\mathbf{PF}(Q)| < \infty$.
□

Definition 1.2.2
Let Q be a finite set, S a finite semigroup, and a partial function $\lambda : Q \times S \to Q$ an *action* of S on Q satisfying two conditions :

$$\text{(i)} \quad \lambda(\lambda(q, s), s_1) = \lambda(q, ss_1) \quad \text{for all} \quad q \in Q; s, s_1 \in S.$$

(ii) $\lambda(q,s) = \lambda(q,s_1), \forall q \in Q$ implies $s = s_1$ where $s, s_1 \in S$.

It is usual to write $\lambda(q,s)$ as qs for $q \in Q, s \in S$.

(Q,S) with this action λ is called a *transformation semigroup.*

Definition 1.2.3

Let $M = (Q, \Sigma, F)$ be a sate machine. we call a transformation semigroup $(Q, \mathbf{S}(M))$ the *transformation semigroup of* M denoting by $\mathbf{TS}(M)$.

Define the *transformation monoid* $\mathbf{TS}(M)^1 = (Q, \mathbf{S}(M)^1)$.

Examples

Let $M = (Q, \Sigma, F)$ be a complete state machine where $Q = \{a, b\}$ and $\Sigma = \{\sigma, \tau, \rho\}$.

F	a	b
σ	a	a
τ	b	a
ρ	a	b

Then $\mathbf{TS}(M)$ is as follows.

λ	a	b
$[\sigma]$	a	a
$[\tau]$	b	a
$[\rho]$	a	b
$[\sigma\tau]$	b	b

And the multiplication table of $\mathbf{S}(M)$ is as follows.

$*$	$[\sigma]$	$[\tau]$	$[\rho]$	$[\sigma\tau]$
$[\sigma]$	$[\sigma]$	$[\sigma\tau]$	$[\sigma]$	$[\sigma\tau]$
$[\tau]$	$[\sigma]$	$[\rho]$	$[\tau]$	$[\sigma\tau]$
$[\rho]$	$[\sigma]$	$[\tau]$	$[\rho]$	$[\sigma\tau]$
$[\sigma\tau]$	$[\sigma]$	$[\sigma]$	$[\sigma\tau]$	$[\sigma\tau]$

As $\mathbf{S}(M)$ is a monoid, $\mathbf{TS}(M)$ is a transformation monoid.

Definition 1.2.4
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines.
Let $\alpha : Q \to Q', \beta : \Sigma \to \Sigma'$ be mappings such that

$$\alpha(qF_\sigma) \subseteq (\alpha(q))F'_{\beta(\sigma)}$$

for any $q \in Q, \sigma \in \Sigma$.

The pair (α, β) is called a *state machine homomorphism* from M to M' and written $(\alpha, \beta) : M \to M'$.

If α and β are both one-one mappings then (α, β) is called a *monomorphism* and if α and β are both onto mappings then (α, β) is called an *epimorphism*. An *isomorphism* of state machines is both a monomorphism and an epimorphism, in this case written $M \cong M'$.

Definition 1.2.5
If $A = (Q, S), A' = (Q', S')$ are transformation semigroups, $f : Q \to Q'$ is a mapping and $g : S \to S'$ a semigroup homomorphism, then the pair (f, g) is said to be a *transformation semigroup homomorphism* from A to A' if

$$f(qs) \subseteq f(q)g(s)$$

for all $q \in Q, s \in S$, written $(f, g) : A \to A'$. Define *monomorphism, epimorphism, isomorphism* of $(f, g) : A \to A'$ in the same way as state machines. Denote isomorphism $(f, g) : A \to A'$ by $A \cong A'$.

Theorem 1.2.6
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be complete state machines. Let $(\alpha, \beta) : M \to M'$ a homomorphism with α onto. Then there exists a homomorphism

$$(f_\alpha, g_\beta) : \mathbf{TS}(M) \to \mathbf{TS}(M').$$

Proof Define $f_\alpha : Q \to Q'$ by $f_\alpha = \alpha$. Put $S = \mathbf{S}(M)$ and $S' = \mathbf{S}(M')$. Let $\forall s \in S$, then there exists $a \in \Sigma^+$ such that $s = [a]$.

Now put $a = \sigma_1\sigma_2 \cdots \sigma_n$, $\sigma_i \in \Sigma$, $i = 1, 2, \ldots, n$
and $\beta(a) = \beta(\sigma_1)\beta(\sigma_2) \cdots \beta(\sigma_n) \in (\Sigma')^+$. Define $g_\beta(s) = [\beta(a)]' \in S'$.

Let $s \in S$. Put $s = [b]$, $b \in \Sigma^+$, $b = \tau_1\tau_2 \cdots \tau_m$, $\tau_i \in \Sigma$, $i = 1, 2, \ldots, m$.
Let $[a] = [b]$, then $qF_a = qF_b$ for $\forall q \in Q$. ... (*)

Since α is onto, there exists $q \in Q$ such that $q' = \alpha(q)$ for $\forall q' \in Q'$. Hence

$$q'F'_{\beta(a)} = (\alpha(q))F'_{\beta(a)},$$
$$q'F'_{\beta(b)} = (\alpha(q))F'_{\beta(b)}.$$

By (*), $\alpha(qF_a) = \alpha(qF_b)$. Since (α, β) is a homomorphism,

$$\alpha(qF_a) = (\alpha(q))F'_{\beta(a)} = (\alpha(q))F'_{\beta(b)}.$$

Hence $q'F'_{\beta(a)} = q'F'_{\beta(b)}$, that is $\beta(a) \sim' \beta(b)$. Therefore $g_\beta : S \to S'$ is well-defined.

Let $q \in Q$ and $s \in S$, where $s = [a]$, $a \in \Sigma^+$. Then

$$\begin{aligned} f_\alpha = \alpha(qs) &= \alpha(qF_a) \\ &= (\alpha(q))F'_{\beta(a)} \\ &= f_\alpha(q)[\beta(a)]' \\ &= f_\alpha(q)g_\beta(s). \end{aligned}$$

Hence (f_α, g_β) is a homomorphism. □

Theorem 1.2.7

Let $A = (Q, S)$ be a transformation semigroup. Then

$$\mathbf{TS}(\mathbf{SM}(A)) \cong A.$$

Proof Put $\mathbf{SM}(A) = (Q, S, F)$ and $K = \langle F(\mathbf{SM}(A))\rangle$. By Proposition 2.1, $K \cong S$, so there exists an isomorphism $\theta : S \to K$, where $\theta(s) = F_s$, $s \in S$.

Now consider a pair of mappings $(1_Q, \theta) : A \to \mathbf{TS}(\mathbf{SM}(A))$, where $\mathbf{TS}(\mathbf{SM}(A)) = (Q, K)$. Then

$$1_Q(qs) = qs = qF_s = 1_Q(q)\theta(s),$$

$\forall q \in Q$, $\forall s \in S$. Hence $(1_Q, \theta)$ is a transformation semigroup isomorphism. □

Theorem 1.2.8

Let $M = (Q, \Sigma, F)$ be a state machine. Then there exists a state machine monomorphism

$$(\alpha, \beta) : M \to \mathbf{SM}(\mathbf{TS}(M)).$$

Proof We can indicate $\mathbf{SM}(\mathbf{TS}(M)) = (Q, \mathbf{S}(M), F)$. Define $\alpha : Q \to Q$ by $\alpha = 1_Q$, and define $\beta : \Sigma \to \mathbf{S}(M)$ by $\beta(\sigma) = [\sigma]$, $\sigma \in \Sigma$. Then clearly α and β are one-one mappings.

Let $q \in Q$ and $\sigma \in \Sigma$. Then

$$\begin{aligned} \alpha(qF_\sigma) &= qF_\sigma \\ &= qF'_{[\sigma]} \\ &= (\alpha(q))F'_{\beta(\sigma)}. \end{aligned}$$

Hence (α, β) is a state machine monomorphism. □

Definition 1.2.9
Let $M = (Q, \Sigma, F)$ be a state machine.
A relation R on Q is called *admissible* if and only if
(i) R is an equivalence relation,
(ii) $(qF_\sigma)R(q_1F_\sigma)$
for $\forall q, q_1 \in Q$, $\forall \sigma \in \Sigma$ such that qRq_1, $qF_\sigma \neq \emptyset$, $q_1F_\sigma \neq \emptyset$.

Proposition 1.2.10
Let $M = (Q, \Sigma, F)$ be a state machine.
Let R be an admissible relation on Q, and $\pi = \{H_i\}_{i \in I}$ a partition on Q induced by R. Define

$$H_iF_\sigma = \{qF_\sigma | q \in H_i\}$$

for $\forall H_i \in \pi$, $\forall \sigma \in \Sigma$. Then there exists $j \in I$ such that $H_iF_\sigma \subseteq H_j$.

Proof Let $q_1 \in H_i$. Then there exists $j \in I$ such that $q_1F_\sigma = q_2 \in H_j$. Let$q \in H_i$. Then qRq_1.

Let $qF_\sigma \neq \emptyset$ and $q_1F_\sigma \neq \emptyset$. Then $(qF_\sigma)R(q_1F_\sigma)$, because R is admissible. Hence $qF_\sigma \in H_j$, so $H_iF_\sigma \subseteq H_j$.

Let $qF_\sigma = \emptyset$ or $q_1F_\sigma = \emptyset$. Then $H_i = \emptyset$. Hence $H_iF_\sigma \subseteq H_j$. □

Definition 1.2.11
Let $M = (Q, \Sigma, F)$ be a state machine.
A partition $\pi = \{H_i\}_{i \in I}$ of Q is called *admissible* if and only if
there exists $j \in I$ such that $H_iF_\sigma \subseteq H_j$ for $\forall i \in I$, $\forall \sigma \in \Sigma$, or $H_iF_\sigma = \emptyset$.

Proposition 1.2.12
Let $M = (Q, \Sigma, F)$ be a complete state machine. Then $\varphi : I \times \Sigma \to I$ is a function.

Proof Since M is complete, $qF_\sigma \in Q$, $qF_\sigma \neq \emptyset$ for $\forall q \in Q$, $\forall \sigma \in \Sigma$. Then $H_iF_\sigma \neq \emptyset$ for $\forall i \in I$. Now assume $H_iF_\sigma \subseteq H_j$ and $H_iF_\sigma \subseteq H_k$, $j \neq k$. Then $H_j \cap H_k \neq \emptyset$. This contradicts with that π is a partition on Q. Hence $j = k$. □

Definition 1.2.13
Let $A = (Q, S)$ be a transformation semigroup.
A relation R on Q is called *admissible* if and only if
(i) R is an equivalence relation,
(ii) $(qs)R(q_1s)$
for $\forall q, q_1 \in Q$, $\forall s \in S$ such that qRq_1, $qs \neq \emptyset$, $q_1s \neq \emptyset$.

Definition 1.2.14
Let $A = (Q, S)$ be a transformation semigroup.
A partition $\pi = \{H_i\}_{i \in I}$ on Q is called *admissible* if and only if there exists $j \in I$ such that $H_is \subseteq H_j$ for $\forall i \in I$, $\forall s \in S$, or $H_is = \emptyset$.

Definition 1.2.15
Let $M = (Q, \Sigma, F)$ be a state machine, and $\pi = \{H_i\}_{i \in I}$ an admissible partition on Q. Put $M/\pi = (Y, \Sigma, G)$ and $Y = \pi$. Let $H_i, H_j \in Y$ and $\sigma \in \Sigma$. Define $H_iG_\sigma = H_j$ if and only if $H_iF_\sigma \subseteq H_j$ for $\forall i, j \in I$, $\forall \sigma \in \Sigma$. And put $H_iG_\sigma = \emptyset$ if and only if $H_iF_\sigma = \emptyset$. Then M/π is called a *quotient state machine* of M with respect to π.

Proposition 1.2.16
Let $M = (Q, \Sigma, F)$ be a state machine. Then $M/\pi = (Y, \Sigma, G)$ is a state machine.

Proof Let $H_k = H_l$ and $G_\sigma = G_\tau$ for $H_k, H_l \in \pi$, $G_\sigma, G_\tau \in G$. Put $H_i = H_kG_\sigma$ and $H_j = H_lG_\tau$.

Suppose $H_kF_\sigma \neq \emptyset$ and $H_lF_\tau \neq \emptyset$. Then there exists just one element $i \in I$ such that $H_kF_\sigma \subseteq H_i$, and just one element $j \in I$ such that $H_lF_\tau \subseteq H_j$. Let $H_i \cap H_j = \emptyset$. Then

$$H_lG_\sigma = H_j \Leftrightarrow H_lF_\sigma \subseteq H_j,$$

$$H_kG_\sigma = H_i \Leftrightarrow H_kF_\sigma \subseteq H_i.$$

Therefore $H_i \cap H_j \neq \emptyset$, so $H_i = H_j$.

Suppose $H_kF_\sigma = \emptyset$ or $H_lF_\tau = \emptyset$. Then $H_kG_\sigma = \emptyset$ or $H_lG_\tau = \emptyset$. Therefore $H_kG_\sigma = H_lG_\tau = \emptyset$. □

Definition 1.2.17
Let $A = (Q, S)$ be a transformation semigroup, and $\pi = \{H_i\}_{i \in I}$ an admissible partition on Q. Put $Y = \pi$. Let $H_i, H_j \in Y$ and $s \in S$. Define $H_i * s = H_j$ if and only if $H_i s \subseteq H_j$. And define $H_i * s = \emptyset$ if and only if $H_i s = \emptyset$.

Proposition 1.2.18
Let $A = (Q, S)$ be a transformation semigroup. Let $i \in I$, and $s, s' \in S$. Define $s \sim s'$ if and only if $H_i * s = H_i * s'$. Then $\sim$ is a congruence relation.

Proof Let $s, s', t \in S$ and $s \sim s'$. As $t \sim t$, $H_i * t = H_i * t$. Let $H_i * t \subseteq H_k$. Then $(H_i * t) * s \subseteq H_k * s$ and $(H_i * t) * s' \subseteq H_k * s'$. As $H_k * s = H_k * s'$, $H_i * (ts) = H_i * (ts')$. Hence $ts \sim ts'$.

Let $s, s', t \in S$ and $s \sim s'$. Then $H_i * s = H_i * s'$. Since there exists $H_j \in \pi$ such that $H_i * s \subseteq H_j$ and $H_i * s' \subseteq H_j$, then

$$(H_i * s) * t \subseteq H_j * t = H_k$$

and

$$(H_i * s') * t \subseteq H_j * t = H_k.$$

As $H_i * (st) = H_k = H_i * (s't)$, $st \sim s't$. Hence $\sim$ is a congruence relation.
□

Corollary 1.2.19
Let $A = (Q, S)$ be a transformation semigroup. Put $S' = S/\sim$ and $Y = \pi$. Let $H_i \in Y$, and $s \in S$. Define $H_i * [s] = H_i * s$. Then $A/\langle\pi\rangle = (Y, S')$ is a transformation semiroup.

Proposition 1.2.20
Let $M = (Q, \Sigma, F)$ be a state machine, and $\mathbf{TS}(M)$ a transformation semigroup induced by M. Then a partition π on Q is admissible with respect to M, if and only if π on Q is admissible with respect to $\mathbf{TS}(M)$.

Proof Let a partition π on Q be admissible with respect to M. Then there exists $j \in I$ such that $H_iF_\sigma \subseteq H_j$ for $\forall i \in I$, $\forall \sigma \in \Sigma$, or $H_iF_\sigma = \emptyset$. Since $[\sigma] \in \mathbf{S}(M)$, then there exists $j \in I$ such that $H_i[\sigma] \subseteq H_j$ for $\forall i \in I$, $\forall [\sigma] \in \mathbf{S}(M)$, or $H_i[\sigma] = \emptyset$.

Let a partition π on Q be admissible with respect to $\mathbf{TS}(M)$. Then there exists $j \in I$ such that $H_i[\sigma] \subseteq H_j$ for $\forall i \in I$, $\forall[\sigma] \in \mathbf{S}(M)$, or $H_i[\sigma] = \emptyset$.

Let $\sigma \in \Sigma$. Then there exists $s \in \mathbf{S}(M)$ such that $[\sigma] = s$. Hence there exists $j \in I$ such that $H_iF_\sigma \subseteq H_j$ for $\forall i \in I$, $\forall \sigma \in \Sigma$, or $H_iF_\sigma = \emptyset$.
□

Proposition 1.2.21
Let $M = (Q, \Sigma, F)$ be a state machine, and $\pi = \{H_i\}_{i \in I}$ an admissible partition on Q. Then

$$\mathbf{TS}(M/\pi) = (\mathbf{TS}(M))/\langle\pi\rangle.$$

Proof

$$\begin{aligned} \mathbf{TS}(M/\pi) &= (Y, \mathbf{S}(M)) \\ &= (Y, \mathbf{S}(\mathbf{S}(M))) \\ &= (\mathbf{TS}(M))/\langle\pi\rangle. \end{aligned}$$

□

Definition 1.2.22
Let $M = (Q, \Sigma, F)$ be a state machine, and $\pi = \{H_i\}_{i \in I}$ an admissible partition on Q. Define $(\alpha^\pi, 1_\Sigma) : M \to M/\pi$ by $\alpha^\pi(q) = H_i$, where $q \in H_i$, $q \in Q$ and $H_i \in \pi$. Then $(\alpha^\pi, 1_\Sigma)$ is called a *natural epimorphism* defined by π.

Proposition 1.2.23
Let $M = (Q, \Sigma, F)$ be a state machine. Then $(\alpha^\pi, 1_\Sigma) : M \to M/\pi$ is an epimorphism.

Proof Let $q \in Q$, $\sigma \in \Sigma$ and $qF_\sigma \neq \emptyset$. Put

$$(\alpha^\pi(q))G_{1_\Sigma(\sigma)} = H_iG_\sigma = H_j.$$

As π is admissible, $qG_\sigma \in H_j$. Put $\alpha^\pi(qF_\sigma) = H_k$. By the definition of α^π, $qF_\sigma \in H_k$. As G_σ and F_σ give the same action on $q \in Q$, $qG_\sigma = qF_\sigma$. Hence $H_j \cap H_k \neq \emptyset$. Since π is a partition on Q, $H_j = H_k$. Hence $\alpha^\pi(qF_\sigma) = (\alpha^\pi(q))G_{1_\Sigma(\sigma)}$.

Let $qF_\sigma = \emptyset$. Then clearly $\alpha^\pi(qF_\sigma) \subseteq (\alpha^\pi(q))G_{1_\Sigma(\sigma)}$. As α^π is surjective clearly, $(\alpha^\pi, 1_\Sigma)$ is a state machine epimorphism. □

Definition 1.2.24

Let $A = (Q, S)$ be a transformation semigroup, and $\pi = \{H_i\}_{i \in I}$ an admissible partition on Q. Define $(f^\pi, g^\pi) : A \to A/\pi$ by $f^\pi(q) = H_i$ and $g^\pi(s) = [s]$, where $q \in H_i$, $q \in Q$ and $H_i \in \pi$, $s \in S$. Then (f^π, g^π) is called a *natural epimorphism* defined by π.

Proposition 1.2.25

Let $A = (Q, S)$ be a transformation semigroup. Then $(f^\pi, g^\pi) : A \to A/\pi$ is an epimorphism.

Proof Let $q \in Q$, $s \in S$. Put $f^\pi(qs) = H_k$. Since

$$f^\pi(q)g^\pi(s) = H_i * [s] = H_j,$$

similarly to the proof of Proposition 2.28,

$$f^\pi(qs) \subseteq f^\pi(q)g^\pi(s).$$

Hence (f^π, g^π) is a transformation semigroup homomorphism. As f^π and g^π are both onto clearly, (f^π, g^π) is a transformation semigroup epimorphism. □

Proposition 1.2.26

Let $M = (Q, \Sigma, F)$ be a state machine. Let $\pi = \{H_i\}_{i \in I}$ and $\pi' = \{K_j\}_{j \in J}$ admissible partitions on Q, where $\pi \leq \pi'$ i.e. there exists $j \in J$ such that $H_i \subseteq K_j$ for $\forall i \in I$. Define $\alpha : \pi \to \pi'$ by $\alpha(H_i) = K_j$. Then $(\alpha, 1_\Sigma) : M/\pi \to M/\pi'$ is an epimorphism.

Proof Put $M/\pi = (Y, \Sigma, G)$ and $M/\pi' = (Y', \Sigma, G')$. Let $H_i \in Y$, $\sigma \in \Sigma$ and $H_iG_\sigma \neq \emptyset$. Put $\alpha(H_iG_\sigma) = \alpha(H_k) = K_l$. Since $H_iG_\sigma \subseteq H_k$ and $H_k \subseteq K_l$, $H_iG_\sigma \subseteq K_l$. Put $((\alpha(H_i))G'_{1_\Sigma(\sigma)} = K_jG_\sigma = K_m$, i.e., $K_jG_\sigma \subseteq K_m$. Since $H_i \subseteq K_j$, $H_iG_\sigma \subseteq K_m$. Hence $K_l \cap K_m \neq \emptyset$, so $K_l = K_m$. Therefore $\alpha(H_iG_\sigma) = (\alpha(H_i))G'_{1_\Sigma(\sigma)}$.

Let $H_iG_\sigma = \emptyset$. Then clearly $\alpha(H_iG_\sigma) = (\alpha(H_i))G'_{1_\Sigma(\sigma)}$. As α and 1_Σ are both onto clearly, $(\alpha, 1_\Sigma)$ is a state machine epimorphism. □

Theorem 1.2.27
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines.
Let $(\alpha, \beta) : M \to M'$ be an epimorphism. Let π_α be an admissible partition defined by α on M, and π an admissible partition on M such that $\pi \leq \pi_\alpha$. Then there exists an epimorphism $(\lambda, \mu) : M/\pi \to M'$.

Furthermore, if $\pi = \pi_\alpha$, then (λ, μ) is an isomorphism.

Proof Put $\pi = \{H_i\}_{i \in I}$ and $\pi_\alpha = \{K_j\}_{j \in J}$. Define $\lambda : \pi \to Q'$ by $\lambda(H_i) = \alpha(q)$, $q \in H_i$.

Let $H_i, H_j \in \pi$ such that $H_i = H_j$. As $\pi \leq \pi_\alpha$, there exists $K_l \in \pi_\alpha$ such that $H_i \subseteq K_l$. Hence $\alpha(q) = \alpha(q_1)$ for $\forall q, q_1 \in H_i$. Therefore λ is well-defined.

As α is onto, there exists $q \in Q$ such that $\alpha(q) = q'$ for $\forall q' \in Q'$. And since there exists $H_i \in \pi$ such that $q \in H_i$ for $\forall q \in Q$, λ is onto.

Define $\mu : \Sigma \to \Sigma'$ by $\mu = \beta$. Then μ is clearly well-defined and onto.

Let $H_i \in \pi$ and $\sigma \in \Sigma$. Then

$$(\lambda(H_i))F'_{\mu(\sigma)} = (\alpha(q))F'_{\beta(\sigma)} \in Q'.$$

Put $H_iG_\sigma = H_j$ and let $q_j \in H_j$. Then

$$\lambda(H_iG_\sigma) = \lambda(H_j) = \alpha(q_j) \in Q'.$$

As $q \in H_i$, $qG_\sigma = qF_\sigma \in H_j$. Therefore $\alpha(qF_\sigma) = \alpha(q_j)$. As (α, β) is an epimorphism, $\alpha(qF_\sigma) \subseteq (\alpha(q))F'_{\beta(\sigma)}$. Therefore $\lambda(H_iG_\sigma) \subseteq (\lambda(H_i))F'_{\mu(\sigma)}$. Hence (λ, μ) is a state machine epimorphism.

Furthermore, consider the case that $\pi = \pi_\alpha$. Let $q'_1, q'_2 \in Q'$ such that $q'_1 = q'_2$. As α is onto, there exists $q_1, q_2 \in Q$ such that $q'_1 = \alpha(q_1)$ and $q'_2 = \alpha(q_2)$. As $q'_1 = q'_2$, $\alpha(q_1) = \alpha(q_2)$. Therefore there exists $H_\alpha \in \pi_\alpha$ such that $q_1, q_2 \in H_\alpha$. As λ is onto, there exists $H_1, H_2 \in \pi$ such that $\lambda(H_1) = \alpha(q_1)$ and $\lambda(H_2) = \alpha(q_2)$, where $q_1 \in H_1$ and $q_2 \in H_2$. As $\pi = \pi_\alpha$, $H_1 = H_\alpha = H_2$. Therefore λ is a one-one mapping. As $\mu : \Sigma \to \Sigma'$ is clearly one-one, (λ, μ) is a state machine isomorphism.
□

Theorem 1.2.28
Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups. Let

$(f, g) : A \to A'$ be an epimorphism. Let π_f be an admissible partition defined by f on A, and π an admissible partition on A such that $\pi \leq \pi_f$. Then there exists an epimorphism $(l, m) : A/\langle\pi\rangle \to A'$.

Furthermore, if $\pi = \pi_f$, then (l, m) is an isomorphism.

Proof Put $\pi = \{H_i\}_{i \in I}$ and $\pi_\alpha = \{K_j\}_{j \in J}$. Define $l : \pi \to Q'$ by $l(H_i) = f(q)$, $q \in H_i$. By the proof of Theorem 2.32, l is clearly well-defined. And as f is onto, l is also onto.

Define $\mu : S/\sim \to S'$ by $m([s]) = g(s_i)$ for $s \in S$, $[s] \in S/\sim$, $s_i \in [s]$.

Let $[s_1], [s_2] \in S/\sim$ such that $[s_1] = [s_2]$. Then $H_i * s_1 = H_i * s_2$ for $\forall H_i \in \pi$. As $\pi \leq \pi_f$, there exists $K_j \in \pi_f$ such that $H_i \subseteq K_j$ for $\forall H_i \in \pi$. Then $K_j * s_1 = K_j * s_2$, that is

$$m([s_1]) = g(s_1) = g(s_2) = m([s_2]).$$

Therefore m is well-defined.

Let $s' \in S'$. As $g : S \to S'$ is onto, there exists $s \in S$ such that $g(s) = s'$. Therefore there exists $s \in S$ such that $m([s]) = g(s) = s'$.

Let $H_i \in \pi$ and $[s] \in S/\sim$. Then

$$l(H_i) * m([s]) = f(q) * g(s)$$

for $q \in H_i$. Let $H_i s \subseteq H_j$ and $q_1 \in H_j$. Put

$$l(H_i * [s]) = l(H_i * s) = l(H_j) = f(q_1).$$

As $qs \in H_j$, $f(qs) = f(q_1)$. As (f, g) is a transformation semigroup epimorphism, $f(qs) \subseteq f(q) * g(s)$. Therefore

$$l(H_i * [s]) \subseteq l(H_i) * m([s]).$$

Hence (l, m) is a transformation semigroup epimorphism.

Furthermore, consider the case that $\pi = \pi_f$. Let $q_1', q_2' \in Q'$ such that $q_1' = q_2'$.

As l is onto, there exists $H_i, H_k \in \pi$, $q_1 \in H_i$ and $q_2 \in H_k$ such that $l(H_i) = f(q_1) = q_1'$ and $l(H_k) = f(q_2) = q_2'$.

As $q_1' = q_2'$, $f(q_1) = f(q_2)$. Therefore there exists $K_j \in \pi_f$ such that $g_1, g_2 \in K_j$. As $\pi = \pi_f$, $H_i = K_j = H_k$. Therefore l is a one-one mapping.

Let $s_1', s_2' \in S'$ such that $s_1' = s_2'$. As m is onto, there exists $[s_1], [s_2] \in S/\sim$ such that

$$m([s_1]) = g(s_1) = s_1'$$

and

$$m([s_2]) = g(s_2) = s_2'.$$

As $g(s_1) = g(s_2)$, $H_i * s_1 = H_i * s_2$ for $\forall H_i \in \pi$. As $\pi = \pi_f$, $H_j * s_1 = H_j * s_2$ for $\forall H_j \in \pi_f$. Therefore $[s_1] = [s_2]$. Hence(l, m) is a transformation semigroup isomorphism. $\square$

1.3. Coverings and Products

The concept of covering is useful for studies on state machines and transformation semigroups. It is found specially effective in studies of of products.

Definition 1.3.1
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines.
Let $\eta : Q' \rightarrow Q$ be a surjective partial function and $\xi : \Sigma \rightarrow \Sigma'$ be a mapping such that

$$\eta(q')F_\alpha \subseteq \eta(q'F'_{\xi(\alpha)})$$

for each $q' \in Q', \alpha \in \Sigma^*$, (η, ξ) is a *covering* of M by M', written $M \leq M'$.

Definition 1.3.2
Let $A = (Q, S)$ and $B = (P, T)$ be transformation semigroups.
Let $\eta : P \rightarrow Q$ be a surjective partial function. Assume there exists $t_s \in T$ for $\forall s \in S$ such that

$$\eta(p) \cdot s \subseteq \eta(p \cdot t_s)$$

for each $p \in P$. Then we call B *covers* A, written $A \leq B$. η is called a *covering* of A by B. And $t_s \in T$ is called a *covering element* for $s \in S$.

Lemma 1.3.3
Let $A = (Q, S)$ and $B = (P, T)$ be transformation semigroups.
And other conditions are the same as Definition 1.3.2. Assume there exists $t_s \in T$ for $\forall s \in S$ such that

$$\eta(p) \cdot s \subseteq \eta(p \cdot t_s)$$

for each $p \in P$. And assume there exists $t_{s'} \in T$ for $\forall s' \in S$ such that

$$\eta(p) \cdot s' \subseteq \eta(p \cdot t_{s'})$$

for each $p \in P$. Then there exists $t_s \cdot t_{s'} \in T$ for $\forall ss' \in S$ such that

$$\eta(p) \cdot ss' \subseteq \eta(p \cdot t_s \cdot t_{s'})$$

for each $p \in P$.

Proof Let $s, s' \in S$ and $p \in P$. Then

$$\begin{aligned} \eta(p)ss' &= (\eta(p)s)s' \\ &\subseteq \eta(pt_s)s' \\ &\subseteq \eta(pt_st_{s'}). \end{aligned}$$

□

Proposition 1.3.4 ([4])
Let (Q_i, S_i) be transformation semigroups for $i = 1, 2, 3$, where $(Q_1, S_1) \leq (Q_2, S_2)$ and $(Q_2, S_2) \leq (Q_3, S_3)$. Then

$$(Q_1, S_1) \leq (Q_3, S_3).$$

Proof Let $(Q_1, S_1) \leq (Q_2, S_2)$. Then there exists a surjective partial function $\eta_1 : Q_2 \to Q_1$, and a $t_{s_1} \in S_2$ for $\forall s_1 \in S_1$, $\forall q_2 \in Q_2$ such that

$$\eta_1(q_2)s_1 \subseteq \eta_1(q_2 t_{s_1}).$$

Let $(Q_2, S_2) \leq (Q_3, S_3)$. Then there exists a surjective partial function $\eta_2 : Q_3 \to Q_2$, and a $t_{s_2} \in S_3$ for $\forall t_{s_1} \in S_2$, $\forall q_3 \in Q_3$ such that

$$\eta_2(q_3)t_{s_1} \subseteq \eta_2(q_3 t_{s_2}).$$

Define $\eta_3 : Q_3 \to Q_1$ by

$$\eta_3(q_3) = \eta_1(\eta_2(q_3))$$

for $q_3 \in Q_3$. Clearly η_3 is a surjective partial function.

Let $s_1 \in S_1$ and $q_3 \in Q_3$. Then there exists a $t_{s_2} \in S_3$ such that

$$\begin{aligned} \eta_3(q_3)s_1 &= \eta_1(\eta_2(q_3))s_1 \\ &\subseteq \eta_1(\eta_2(q_3)t_{s_1}) \\ &\subseteq \eta_1(\eta_2(q_3 t_{s_2})) \\ &= \eta_3(q_3 t_{s_2}). \end{aligned}$$

Thus $(Q_1, S_1) \leq (Q_3, S_3)$. □

Theorem 1.3.5
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines such that $M \leq M'$. Then

$$\mathbf{TS}(M) \leq \mathbf{TS}(M').$$

Proof Since $M \leq M'$, there exist a surjective partial function $\eta : Q' \to Q$ and a function $\xi : \Sigma \to \Sigma'$ such that

$$\eta(q')F_\alpha \subseteq \eta(q'F'_{\xi(\alpha)})$$

for $\forall q' \in Q'$ and $\forall \alpha \in \Sigma^*$.

Put $\mathbf{TS}(M) = (Q, \mathbf{S}(M)) = (Q, S)$ and $\mathbf{TS}(M') = (Q', \mathbf{S}(M')) = (Q', S')$. Let $s \in S$. Then there exists $a \in \Sigma^*$ such that $s = [a]$. Put $t_s = [\xi(a)] \in S'$. Then

$$\begin{aligned} \eta(q')s &= \eta(q')F_a \\ &\subseteq \eta(q'F'_{\xi(a)}) \\ &= \eta(q't_s) \end{aligned}$$

for $\forall q' \in Q'$. Therefore there exists $t_s \in S'$ such that

$$\eta(q')s \subseteq \eta(q't_s)$$

for $\forall s \in S$ and $\forall q' \in Q'$. Hence $\mathbf{TS}(M) \leq \mathbf{TS}(M')$. □

Definition 1.3.6
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be complete state machines.

(1) Define their *restricted direct product* :

$$M \wedge M' = (Q \times Q', \Sigma, F \wedge F'),$$

in the special case where $\Sigma = \Sigma'$ only, by

$$(F \wedge F')((q, q'), \sigma) = (F(q, \sigma), F'(q', \sigma))$$

for $\sigma \in \Sigma, (q, q') \in Q \times Q'$.

(2) Define the (full) *direct product* of M and M', $M \times M' = (Q \times Q', \Sigma \times \Sigma', F \times F')$ where

$$(F \times F')((q, q'), (\sigma, \sigma')) = (F(q, \sigma), F'(q', \sigma'))$$

for $\sigma \in \Sigma, \sigma' \in \Sigma', (q, q') \in Q \times Q'$.

(3) Define the *cascade product* of M and M' with *respect to* $\omega : Q' \times \Sigma' \to \Sigma$ by

$$M\omega M' = (Q \times Q', \Sigma', F^{\omega})$$

where

$$F^{\omega}((q, q'), \sigma') = (F(q, \omega(q', \sigma')), F'(q', \sigma')),$$

for $\sigma' \in \Sigma', (q, q') \in Q \times Q'$.

(4) Define the *wreath product*, $M \circ M'$, of M and M' where

$$M \circ M' = (Q \times Q', \Sigma^{Q'} \times \Sigma', F^{\circ})$$

and

$$F^{\circ}((q, q'), (f, \sigma')) = (F(q, f(q')), F'(q', \sigma'))$$

for $\sigma' \in \Sigma', f \in \Sigma^{Q'}, (q, q') \in Q \times Q'$.

Proposition 1.3.7

Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Let $\alpha \in \Sigma^+$, $[\alpha]_\wedge \in \mathbf{S}(M \wedge M')$, $[\alpha] \in \mathbf{S}(M)$ and $[\alpha]' \in \mathbf{S}(M')$. Then

$$[\alpha]_\wedge = [\alpha] \cap [\alpha]'.$$

Proof Let $\beta \in \Sigma^+$ such that $\beta \in [\alpha]_\wedge$. Then

$$(F \wedge F')((q, q'), \beta) = (F \wedge F')((q, q'), \alpha)$$

for $\forall (q, q') \in Q \times Q'$.

$\Leftrightarrow$

$$(qF_\beta, q'F'_\beta) = (qF_\alpha, q'F'_\alpha)$$

for $\forall q \in Q$ and $\forall q' \in Q'$.

$\Leftrightarrow$

$$qF_\beta = qF_\alpha$$

for $\forall q \in Q$, and

$$q'F'_\beta = q'F'_\alpha$$

for $\forall q' \in Q'$.

$\Leftrightarrow$

$$[\alpha]_\wedge \subseteq [\alpha] \cap [\alpha]'$$

and

$$[\alpha]_\wedge \supseteq [\alpha] \cap [\alpha]'.$$

Hence

$$[\alpha]_\wedge = [\alpha] \cap [\alpha]'.$$

□

Definition 1.3.8
Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups.
Define their *restricted direct product* with epimorphisms
$\theta : \Sigma^+ \to S, \theta' : \Sigma^+ \to S'$ for a suitable free semigroup Σ^+ :

$$A \wedge A' = (Q \times Q', T),$$

where $T = \Sigma^+/(R_\theta \cap R_{\theta'})$, R_θ and $R_{\theta'}$ are the equivalence classes induced by θ and θ' respectively, and the action is given by :

$$(q, q')[\alpha]_\wedge = (q\theta(\alpha), q'\theta'(\alpha))$$

for $(q, q') \in Q \times Q'$ and $[\alpha]_\wedge \in T$.

Proposition 1.3.9
Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups. Then the action of T on $Q \times Q'$ defined by Definition 3.8 is well-defined.

Proof Consider the action $\lambda : (Q \times Q') \times T \to Q \times Q'$. Let $(q_1, q_1'), (q_2, q_2') \in Q \times Q'$ and $[\alpha_1]_\wedge, [\alpha_2]_\wedge \in T$ such that $q_1 = q_2$, $q_1' = q_2'$ and $[\alpha_1]_\wedge = [\alpha_2]_\wedge$. Then

$$\begin{aligned}
(q_1, q_1')[\alpha_1]_\wedge &= (q_1\theta(\alpha_1), q_1'\theta'(\alpha_1)) \\
&= (q_1\theta(\alpha_2), q_1'\theta'(\alpha_2)) \\
&= (q_2\theta(\alpha_2), q_2'\theta'(\alpha_2)) \\
&= (q_2, q_2')[\alpha_2]_\wedge.
\end{aligned}$$

Therefore λ is a partial function.
Let $(q, q') \in Q \times Q'$ and $[\alpha]_\wedge, [\beta]_\wedge \in T$. Then

$$\begin{aligned}
((q, q')[\alpha]_\wedge)[\beta]_\wedge &= (q\theta(\alpha), q'\theta'(\alpha))[\beta]_\wedge \\
&= ((q\theta(\alpha))\theta(\beta), (q'\theta'(\alpha))\theta'(\beta)) \\
&= (q(\theta(\alpha)\theta(\beta)), q'(\theta'(\alpha)\theta'(\beta))) \\
&= (q, q')([\alpha]_\wedge[\beta]_\wedge).
\end{aligned}$$

Let $(q, q') \in Q \times Q'$. Then

$$(q, q')[\alpha]_\wedge = (q, q')[\beta]_\wedge$$

$\Leftrightarrow$

$$(q\theta(\alpha), q'\theta'(\alpha)) = (q\theta(\beta), q'\theta'(\beta))$$

$\Leftrightarrow$

$$q\theta(\alpha) = q\theta(\beta), \quad q'\theta'(\alpha) = q'\theta'(\beta)$$

$\Leftrightarrow$

$$[\alpha]_\theta = [\beta]_\theta, \quad [\alpha]_{\theta'} = [\beta]_{\theta'}$$

$\Leftrightarrow$

$$[\alpha]_\wedge = [\beta]_\wedge.$$

Hence the action of T on $Q \times Q'$ is well-defined. □

Definition 1.3.10
Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups.
Define the (full) *direct product* of A and A',

$$A \times A' = (Q \times Q', S \times S')$$

where the action is given by :

$$(q, q')(s, s') = (qs, q's')$$

for $(s, s') \in S \times S', (q, q') \in Q \times Q'$.

Definition 1.3.11
Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups.
Define the *wreath product*, $A \circ A'$, of A and A' where

$$A \circ A' = (Q \times Q', S^{Q'} \times S')$$

and the action is defined by

$$(q, q')(f, s') = (q(f(q')), q's')$$

for $s' \in S', f \in S^{Q'}, (q, q') \in Q \times Q'$.

Theorem 1.3.12
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M \wedge M') = \mathbf{TS}(M) \wedge \mathbf{TS}(M')$$

for suitable epimorphisms $\theta : \Sigma^+ \to \mathbf{S}(M), \theta' : \Sigma^+ \to \mathbf{S}(M')$.

Proof We can put $\mathbf{TS}(M \wedge M') = (Q \times Q', \mathbf{S}(M \wedge M'))$, where $\mathbf{S}(M \wedge M') = \Sigma^+ / \sim \cap \sim'$. And we can put $\mathbf{TS}(M) \wedge \mathbf{TS}(M') = (Q \times Q', T)$, where $T = \Sigma^+/(R_\theta \cap R_{\theta'})$. Define θ by $\theta(\alpha) = [\alpha]$ for $\alpha \in \Sigma^+$, and θ' by $\theta'(\beta) = [\beta]'$ for $\beta \in \Sigma^+$. Then $T = \Sigma^+ / \sim \cap \sim'$. □

1.4. Direct products and Wreath products

I will study about direct products, wreath products and their relations in more detail. All state machines and transformation semigroups will be assumed to be complete through this section.

Theorem 1.4.1
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M \times M') \leq \mathbf{TS}(M) \times \mathbf{TS}(M').$$

Proof Put $[(\alpha, \beta)]_\times \in \mathbf{S}(M \times M')$, where $(\alpha, \beta) \in (\Sigma \times \Sigma')^+$. Then

$$(\alpha, \beta), (\alpha_1, \beta_1) \in [(\alpha, \beta)]_\times$$

$\Leftrightarrow$

$$(\alpha, \beta) \sim_\times (\alpha_1, \beta_1)$$

$\Leftrightarrow$

$$(F \times F')_{(\alpha,\beta)} = (F \times F')_{(\alpha_1,\beta_1)}$$

$\Leftrightarrow$

$$F_\alpha = F_{\alpha_1}, \quad F'_\beta = F'_{\beta_1}$$

$\Leftrightarrow$

$$\alpha \sim \alpha_1, \quad \beta \sim' \beta_1$$

$\Leftrightarrow$

$$\alpha_1 \in [\alpha], \quad \beta_1 \in [\beta]'.$$

Define $g : \mathbf{S}(M \times M') \to \mathbf{S}(M) \times \mathbf{S}(M')$ by

$$g([(\alpha, \beta)]_\times) = ([\alpha], [\beta]')$$

for $[(\alpha,\beta)]_\times \in \mathbf{S}(M \times M')$.

Let $[(\alpha_1,\beta_1)]_\times, [(\alpha_2,\beta_2)]_\times \in \mathbf{S}(M\times M')$ such that $[(\alpha_1,\beta_1)]_\times = [(\alpha_2,\beta_2)]_\times$ Then

$$[\alpha_1] = [\alpha_2]', \quad [\beta_1] = [\beta_2]'.$$

Therefore $g([(\alpha_1,\beta_1)]_\times) = g([(\alpha_2,\beta_2)]_\times)$. As it is clear that $\mathbf{D}(g) = \mathbf{S}(M \times M')$, g is a function.

Let $g([(\alpha_1,\beta_1)]_\times) = g([(\alpha_2,\beta_2)]_\times)$. Then $([\alpha_1],[\beta_1]') = ([\alpha_2],[\beta_2]')$. Therefore $\alpha_1 \in [\alpha_2]$, $\beta_1 \in [\beta_2]$. $\Leftrightarrow$

$$[(\alpha_1,\beta_1)]_\times = [(\alpha_2,\beta_2)]_\times.$$

Hence g is one-one.

Let $[(\alpha_1,\beta_1)]_\times, [(\alpha_2,\beta_2)]_\times \in \mathbf{S}(M \times M')$. Then

$$\begin{aligned}
g([(\alpha_1,\beta_1)]_\times[(\alpha_2,\beta_2)]_\times) &= g([(\alpha_1\alpha_2,\beta_1\beta_2)]_\times) \\
&= ([\alpha_1\alpha_2],[\beta_1\beta_2]') \\
&= ([\alpha_1][\alpha_2],[\beta_1]'[\beta_2]') \\
&= ([\alpha_1],[\beta_1]')([\alpha_2],[\beta_2]') \\
&= g([(\alpha_1,\beta_1)]_\times)g([(\alpha_2,\beta_2)]_\times).
\end{aligned}$$

Hence g is a semigroup monomorphism.

Now consider a covering of $\mathbf{TS}(M \times M') = (Q \times Q', \mathbf{S}(M \times M'))$ by $\mathbf{TS}(M)\times\mathbf{TS}(M') = (Q\times Q', \mathbf{S}(M)\times\mathbf{S}(M'))$. Define $\eta : Q\times Q' \to Q\times Q'$ by $\eta = 1_{Q\times Q'}$. Then η is a surjective partial function clearly. Let $[(\alpha,\beta)]_\times \in \mathbf{S}(M \times M')$ and $(q,q') \in Q \times Q'$. Then

$$\begin{aligned}
\eta((q,q'))[(\alpha,\beta)]_\times &= (q,q')[(\alpha,\beta)]_\times \\
&= (q\alpha, q'\beta) \\
&= (q,q')([\alpha],[\beta]') \\
&= \eta((q,q')([\alpha],[\beta]')).
\end{aligned}$$

Since g is a semigroup monomorphism, there exists $([\alpha],[\beta]') \in \mathbf{S}(M) \times \mathbf{S}(M')$ for $\forall[(\alpha,\beta)] \in \mathbf{S}(M \times M')$. Hence $\mathbf{TS}(M \times M') \leq \mathbf{TS}(M) \times \mathbf{TS}(M')$. $\square$

Theorem 1.4.2

Let $A = (Q,S)$ and $A' = (Q',S')$ be transformation semigroups. Let Σ be a set such that $0 < |\Sigma| < \infty$. And let $\theta : \Sigma^+ \to S$ and $\theta' : \Sigma^+ \to S'$ be semigroup homomorphisms. Then

$$A \wedge A' \leq A \times A'.$$

Proof We can indicate $A \wedge A' = (Q \times Q', T)$, $T = \Sigma^+/(R_\theta \cap R_{\theta'})$, and $A \times A' = (Q \times Q', S \times S')$.

Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q \times Q'}$. Then η is a surjective partial function clearly. Let $[\alpha]_\wedge \in T$. As $\alpha \in \Sigma^+$, there exist $\theta(\alpha) \in S$ and $\theta'(\alpha) \in S'$. Therefore there exists $(\theta(\alpha), \theta'(\alpha)) \in S \times S'$. Let $(q, q') \in Q \times Q'$. Then

$$\begin{aligned} (q, q')[\alpha]_\wedge &= (q\theta(\alpha), q'\theta'(\alpha)) \\ &= (q, q')(\theta(\alpha), \theta'(\alpha)). \end{aligned}$$

Hence $A \wedge A' \leq A \times A'$. □

Proposition 1.4.3

Let $A = (Q, S)$ and $A' = (Q', S')$ be transformation semigroups. Then wreath product of A and A' $A \circ A' = (Q \times Q', S^{Q'} \times S')$ is a transformation semigroup.

Proof Define $f * f_1 \in S^{Q'}$ by

$$f * f_1(q') = f(q')f_1(q's')$$

for $f, f_1 \in S^{Q'}$, $\forall q' \in Q'$ and $s' \in S'$.

Let $q_1', q_2' \in Q'$ such that $q_1' = q_2'$. Then

$$\begin{aligned} f * f_1(q_1') &= f(q_1')f_1(q_1's') \\ &= f(q_2')f_1(q_2's') \\ &= f * f_1(q_2'). \end{aligned}$$

Hence $f * f_1 \in S^{Q'}$.

Define a multiplication in $S^{Q'} \times S'$ by

$$(f, s')(f_1, s_1') = (f * f_1, s's_1')$$

for $\forall (f, s'), (f_1, s_1') \in S^{Q'} \times S'$. Clearly the multiplication is well-defined.

Let $(f_1, s_1'), (f_2, s_2'), (f_3, s_3') \in S^{Q'} \times S'$ and $q' \in Q'$. Then

$$\begin{aligned} ((f_1 * f_2) * f_3)(q') &= ((f_1 * f_2)(q')f_3(q'(s_1's_2')) \\ &= (f_1(q')f_2(q's_1')f_3(q'(s_1's_2')) \\ &= f_1(q')(f_2(q's_1')f_3((q's_1')s_2')) \\ &= f_1(q')(f_2 * f_3)(q's_1') \\ &= (f_1 * (f_2 * f_3))(q'). \end{aligned}$$

Hence $S^{Q'} \times S'$ is a semigroup.

Define an action of $S^{Q'} \times S'$ on $Q \times Q'$ by

$$(q,q')(f,s') = (q(f(q')),q's')$$

for $\forall(q,q') \in Q \times Q'$ and $\forall(f,s') \in S^{Q'} \times S'$.

Let $(q_1,q_1'),(q_2,q_2') \in Q \times Q'$ such that $(q_1,q_1') = (q_2,q_2')$, and $(f_1,s_1'),(f_2,s_2') \in S^{Q'} \times S'$ such that $(f_1,s_1') = (f_2,s_2')$. Then

$$\begin{aligned}(q_1,q_1')(f_1,s_1') &= (q_1(f_1(q_1')),q_1's_1') \\ &= (q_2(f_2(q_2')),q_2's_2') \\ &= (q_2,q_2')(f_2,s_2').\end{aligned}$$

Therefore the action is one-valued.

Let $(q,q') \in Q \times Q'$ and $(f,s'),(f_1,s_1') \in S^{Q'} \times S'$. Then

$$\begin{aligned}((q,q')(f,s'))(f_1,s_1') &= (q(f(q')),q's')(f_1,s_1') \\ &= ((q(f(q')))(f_1(q's')),(q's')s_1') \\ &= (q(f(q')f_1(q's')),q'(s's_1')) \\ &= (q((f*f_1)(q')),q'(s's_1')) \\ &= (q,q')(f*f_1,s's_1') \\ &= (q,q')((f,s')(f_1,s_1')).\end{aligned}$$

Let $(q,q') \in Q \times Q'$ and $(f,s'),(f_1,s_1') \in S^{Q'} \times S'$. Then

$$(q,q')(f,s') = (q,q')(f_1,s_1')$$

$\Leftrightarrow$

$$(q(f(q')),q's') = (q(f_1(q')),q's_1')$$

$\Leftrightarrow$

$$q(f(q')) = q(f_1(q')), \quad q's' = q's_1'$$

$\Rightarrow$

$$f(q') = f_1(q'), \quad s' = s_1'$$

$\Leftrightarrow$

$$f = f_1, \quad s' = s_1'$$

$\Leftrightarrow$

$$(f,s') = (f_1,s_1').$$

Hence the action is well defined and faithful. □

Theorem 1.4.4

Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines and $\omega : Q' \times \Sigma' \to \Sigma$ a mapping. Then

$$\text{(i)} \quad \mathbf{TS}(M\omega M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M')$$

$$\text{(ii)} \quad \mathbf{TS}(M \circ M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M').$$

Proof (i)

Define $\omega^+ : Q' \times (\Sigma')^+ \to \Sigma^+$ inductively by

$$\begin{cases} \omega^+(q', \sigma'\alpha') = \omega(q', \sigma')\omega^+(q'F'_{\sigma'}, \alpha') \\ \omega^+(q', \sigma') = \omega(q', \sigma') \end{cases}$$

for $\sigma' \in \Sigma'$, $\alpha' \in (\Sigma')^+$, $q' \in Q'$.

Let $(q'_1, \alpha'_1), (q'_2, \alpha'_2) \in Q' \times (\Sigma')^+$ such that $(q'_1, \alpha'_1) = (q'_2, \alpha'_2)$. Then we can put $\alpha'_1 = \sigma'_1\sigma'_2 \cdots \sigma'_n$, $\alpha'_2 = \tau'_1\tau'_2 \cdots \tau'_n$, $\sigma'_i = \tau'_i$, $\sigma'_i, \tau'_i \in \Sigma'$. And then

$$\begin{aligned}
\omega^+(q'_1, \alpha'_1) &= \omega(q'_, \sigma'_1)\omega^+(q'_1F'_{\sigma'_1}, \sigma'_2 \cdots \sigma'_n) \\
&= \omega(q'_, \sigma'_1)\omega^+(q'_1F'_{\sigma'_1}, \sigma'_2)\omega^+(q'_1F'_{\sigma'_1}F'_{\sigma'_2}, \sigma'_3 \cdots \sigma'_n) \\
&= \omega(q'_, \sigma'_1)\omega^+(q'_1F'_{\sigma'_1}, \sigma'_2)\omega^+(q'_1F'_{\sigma'_1\sigma'_2}, \sigma'_3 \cdots \sigma'_n) \\
&= \omega(q'_, \sigma'_1)\omega^+(q'_1F'_{\sigma'_1}, \sigma'_2)\omega(q'_1F'_{\sigma'_1\sigma'_2}, \sigma'_3) \\
&\quad\ \omega(q'_1F'_{\sigma'_1\sigma'_2\sigma'_3}, \sigma'_4) \cdots \omega(q'_1F'_{\sigma'_1 \cdots \sigma'_{n-1}}, \sigma'_n) \\
&= \omega(q'_, \sigma'_1)\omega^+(q'_1F'_{\tau'_1}, \sigma'_2)\omega(q'_1F'_{\tau'_1\tau'_2}, \sigma'_3) \\
&\quad\ \omega(q'_1F'_{\tau'_1\tau'_2\tau'_3}, \sigma'_4) \cdots \omega(q'_1F'_{\tau'_1 \cdots \tau'_{n-1}}, \sigma'_n) \\
&= \omega(q'_, \tau'_1)\omega^+(q'_1F'_{\tau'_1}, \tau'_2)\omega(q'_1F'_{\tau'_1\tau'_2}, \tau'_3) \\
&\quad\ \omega(q'_1F'_{\tau'_1\tau'_2\tau'_3}, \tau'_4) \cdots \omega(q'_1F'_{\tau'_1 \cdots \tau'_{n-1}}, \tau'_n) \\
&= \omega^+(q'_2, \alpha'_2).
\end{aligned}$$

Therefore ω^+ is well-defined, and as both ω and F' are mappings, ω^+ is a mapping.

So we can define $F^{\omega}_{\alpha'} : Q \times Q' \to Q \times Q'$ by

$$(q, q')F^{\omega}_{\alpha'} = (qF_{\omega^+(q', \alpha')}, q'F'_{\alpha'})$$

for $\forall \alpha' \in (\Sigma')^+$.

Define $\omega^+_{\alpha'} : Q' \to \Sigma^+$ by

$$\omega^+_{\alpha'}(q') = \omega^+(q', \alpha')$$

for $\forall q' \in Q'$. Clearly $\omega^+_{\alpha'}$ is a mapping. So we can define a function $f_{\alpha'} : Q' \to \mathbf{S}(M)$ by

$$f_{\alpha'}(q') = [\omega^+_{\alpha'}(q')]$$

for $\forall q' \in Q'$.

Define $\varphi : \mathbf{S}(M\omega M') \to \mathbf{S}(M) \times \mathbf{S}(M')$ by

$$\varphi([\alpha']_\omega) = (f_{\alpha'}, [\alpha']')$$

for $\forall [\alpha']_\omega \in \mathbf{S}(M\omega M')$.

Let $[\alpha']_\omega, [\beta']_\omega \in \mathbf{S}(M\omega M')$ such that $[\alpha']_\omega = [\beta']_\omega$ and let $(q, q') \in Q \times Q'$. then

$$(q, q')F^\omega_{\alpha'} = (q, q')F^\omega_{\beta'}$$

$\Leftrightarrow$

$$(qF_{\omega^+(q',\alpha')}, q'F'_{\alpha'}) = (qF_{\omega^+(q',\beta')}, q'F'_{\beta'})$$

$\Leftrightarrow$

$$qF_{\omega^+(q',\alpha')} = qF_{\omega^+(q',\beta')}, \quad q'F'_{\alpha'} = q'F'_{\beta'}$$

$\Leftrightarrow$

$$F_{\omega^+(q',\alpha')} = F_{\omega^+(q',\beta')}, \quad F'_{\alpha'} = F'_{\beta'}$$

$\Leftrightarrow$

$$f_{\alpha'} = f_{\beta'}, \quad [\alpha']' = [\beta']'$$

$\Leftrightarrow$

$$(f_{\alpha'}, [\alpha']') = (f_{\beta'}, [\beta']').$$

Hence φ is a partial function.

Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q\times Q'}$. Clearly η is a surjective partial function.

Let $[\alpha']_\omega \in \mathbf{S}(M\omega M')$ and $(q, q') \in Q \times Q'$. Then

$$\begin{aligned}
\eta((q, q'))[\alpha']_\omega &= (q, q')[\alpha']_\omega \\
&= (q, q')F^\omega_{\alpha'} \\
&= (qF_{\omega^+(q',\alpha')}, q'F'_{\alpha'}) \\
&= (q[\omega^+(q', \alpha')], q'[\alpha']') \\
&= (q[\omega^+_{\alpha'}(q')], q'[\alpha']')
\end{aligned}$$

$$
\begin{aligned}
&= (qf_{\alpha'}(q'), q'[\alpha']') \\
&= (q, q')(f_{\alpha'}(q'), [\alpha']') \\
&\subseteq (q, q')(f_{\alpha'}, [\alpha']') \\
&= \eta((q, q')(f_{\alpha'}, [\alpha']')).
\end{aligned}
$$

Thus $\mathbf{TS}(M\omega M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M')$.

(ii)

Define $\varphi : \mathbf{S}(M \circ M') \to \mathbf{S}(M)^{Q'} \times \mathbf{S}(M')$ by

$$\varphi([(f, \alpha')]^{\circ}) = ([f], [\alpha']')$$

for $\forall [(f, \alpha')]^{\circ} \in \mathbf{S}(M \circ M')$.

Let $[(f_1, \alpha'_1)]^{\circ}, [(f_2, \alpha'_2)]^{\circ} \in \mathbf{S}(M \circ M')$ such that $[(f_1, \alpha'_1)]^{\circ} = [(f_2, \alpha'_2)]^{\circ}$ and let $(q, q') \in Q \times Q'$. Then

$$F^{\circ}((q, q'), [(f_1, \alpha'_1)]^{\circ}) = F^{\circ}((q, q'), [(f_2, \alpha'_2)]^{\circ})$$

$\Leftrightarrow$

$$(F(q, f_1(q')), F'(q', \alpha'_1)) = (F(q, f_2(q')), F'(q', \alpha'_2))$$

$\Leftrightarrow$

$$F(q, f_1(q')) = F(q, f_2(q')), \quad F'(q', \alpha'_1) = F'(q', \alpha'_2)$$

$\Leftrightarrow$

$$F_{f_1(q')} = F_{f_2(q')}, \quad [\alpha'_1]' = [\alpha'_2]'$$

$\Leftrightarrow$

$$[f_1] = [f_2], \quad [\alpha'_1]' = [\alpha'_2]'$$

$\Leftrightarrow$

$$([f_1], [\alpha'_1]') = ([f_2], [\alpha'_2]').$$

Therefore φ is a partial function.

Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q \times Q'}$. Clearly η is a surjective partial function.

Let $[(f, \alpha')]^{\circ} \in \mathbf{S}(M \circ M')$ and $(q, q') \in Q \times Q'$. Then

$$
\begin{aligned}
\eta((q, q'))[(f, \alpha')]^{\circ} &= (q, q')[(f, \alpha')]^{\circ} \\
&= (F(q, f(q')), F'(q', \alpha')) \\
&= (F(q, [f(q')]), F'(q', [\alpha']')) \\
&= (q, q')([f(q')], [\alpha']') \\
&= (q, q')([f], [\alpha']') \\
&= \eta((q, q')([f], [\alpha']')).
\end{aligned}
$$

Thus $\mathbf{TS}(M \circ M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M')$. $\square$

Proposition 1.4.5 ([2])
Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M) \times \mathbf{TS}(M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M').$$

Proof Define $\varphi : \mathbf{S}(M) \times \mathbf{S}(M') \to \mathbf{S}(M)^{Q'} \times \mathbf{S}(M')$ by

$$\varphi(([\alpha], [\beta]')) = (f_{[\alpha]}, [\beta]')$$

for $([\alpha], [\beta]') \in \mathbf{S}(M) \times \mathbf{S}(M')$, where $f_{[\alpha]} : Q' \to \mathbf{S}(M)$ is defined by

$$f_{[\alpha]}(q') = [\alpha]$$

for $\forall q' \in Q'$. Then $f_{[\alpha]}$ is a function clearly.
Let $([\alpha_1], [\beta_1]'), ([\alpha_2], [\beta_2]') \in \mathbf{S}(M) \times \mathbf{S}(M')$ such that $([\alpha_1], [\beta_1]') = ([\alpha_2], [\beta_2]')$. Then

$$([\alpha_1], [\beta_1]') = ([\alpha_2], [\beta_2]')$$

$\Leftrightarrow$

$$[\alpha_1] = [\alpha_2], \quad [\beta_1]' = [\beta_2]'.$$

Therefore

$$\begin{aligned} \varphi(([\alpha_1], [\beta_1]')) &= (f_{[\alpha_1]}, [\beta_1]') \\ &= (f_{[\alpha_2]}, [\beta_2]') \\ &= \varphi(([\alpha_2], [\beta_2]')). \end{aligned}$$

Hence φ is well-defined. As it is clear that $\mathbf{D}(\varphi) = \mathbf{S}(M) \times \mathbf{S}(M')$, φ is a function.
Let $\varphi(([\alpha_1], [\beta_1]')) = \varphi(([\alpha_2], [\beta_2]'))$. Then

$$(f_{[\alpha_1]}, [\beta_1]') = (f_{[\alpha_2]}, [\beta_2]')$$

$\Leftrightarrow$

$$f_{[\alpha_1]} = f_{[\alpha_2]}, \quad [\beta_1]' = [\beta_2]'$$

$\Leftrightarrow$

$$[\alpha_1] = [\alpha_2], \quad [\beta_1]' = [\beta_2]'$$

$\Leftrightarrow$

$$([\alpha_1],[\beta_1]') = ([\alpha_2],[\beta_2]').$$

Hence φ is one-one.

Let $([\alpha_1],[\beta_1]'), ([\alpha_2],[\beta_2]') \in \mathbf{S}(M) \times \mathbf{S}(M')$. Then

$$\begin{aligned}
\varphi(([\alpha_1],[\beta_1]')([\alpha_2],[\beta_2]')) &= \varphi(([\alpha_1][\alpha_2],[\beta_1]'[\beta_2]')) \\
&= \varphi(([\alpha_1\alpha_2],[\beta_1\beta_2]')) \\
&= (f_{[\alpha_1\alpha_2]},[\beta_1\beta_2]') \\
&= (f_{[\alpha_1][\alpha_2]},[\beta_1]'[\beta_2]') \\
&= (f_{[\alpha_1]} * f_{[\alpha_2]},[\beta_1]'[\beta_2]') \\
&= (f_{[\alpha_1]},[\beta_1]')(f_{[\alpha_2]},[\beta_2]') \\
&= \varphi(([\alpha_1],[\beta_1]'))\varphi(([\alpha_2],[\beta_2]')).
\end{aligned}$$

Hence φ is a semigroup monomorphism.

Now consider a covering of $\mathbf{TS}(M) \times \mathbf{TS}(M')$ by $\mathbf{TS}(M) \circ \mathbf{TS}(M')$. Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q\times Q'}$. Then η is a surjective partial function clearly.

Let $([\alpha],[\beta]') \in \mathbf{S}(M) \times \mathbf{S}(M')$ and $(q,q') \in Q \times Q'$. Then

$$\begin{aligned}
\eta((q,q'))([\alpha],[\beta]') &= (q,q')([\alpha],[\beta]') \\
&= (q[\alpha],q'[\beta]') \\
&= (q(f_{[\alpha]}(q')),q'[\beta]') \\
&= (q,q')(f_{[\alpha]},[\beta]') \\
&= \eta((q,q')(f_{[\alpha]},[\beta]')).
\end{aligned}$$

Hence $\mathbf{TS}(M) \times \mathbf{TS}(M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M')$. $\square$

2. Switchboard state machines

2.1. Switchboard state machines and switchboard transformation semigroups

First we shall introduce some concepts.

Definition 2.1.1. Let $M = (Q, \Sigma, F)$ be a state machine.
If $pF_\sigma = q \Rightarrow qF_\sigma = p$ for each $p, q \in Q, \sigma \in \Sigma, \sigma \neq \emptyset$, then M is called *switching.*
If $qF_{\sigma\tau} = qF_{\tau\sigma}$ for each $q \in Q, \sigma, \tau \in \Sigma$, then M is called *commutative*([5]).
If M is *switching* and *commutative*, then M is called a *switchboard state machine.*

Definition 2.1.2. Let $A = (Q, S)$ be a transfomation semigroup. If $ps = q \Rightarrow qs = p$ for each $p, q \in Q, s \in S$, then A is called *switching.*
If $q(st) = q(ts)$ for each $q \in Q, s, t \in S$, then A is called *commutative.*
If A is *switching* and *commutative*, then A is called a *switchboard transformation semigroup.*

Example 2.1.3. (1) Consider the state machine
$M = (Q, \Sigma, F)$, $Q = \{1, 2, 3, 4\}$,
$\Sigma = \{\sigma, \tau\}$ defined by the following action table, then M is a switchboard state machine.

F	1	2	3	4
σ	3	4	1	2
τ	2	1	4	3

(2) Consider the state machine
$M' = (Q', \Sigma', F')$, $Q' = \{0, 1, 2\}$,
$\Sigma' = \{\sigma', \tau'\}$ defined by the following action table, then M is a switching state machine, but not a commutative state machine, and so not a switchboard state machine because $\sigma'\tau' \neq \tau'\sigma'$.

F'	0	1	2
σ'	0	2	1
τ'	1	0	2

(3) Consider the state machine $M = (Q, \Sigma, F)$ given by

F	0	1	2
σ	2	0	1
τ	1	2	0

This is commutative because $\sigma\tau = \tau\sigma$, but not switching.

Proposition 2.1.4. Let $M = (Q, \Sigma, F)$ be a switchboard state machine. Then $\mathbf{TS}(M)$ is a switchboard transformation semigroup.

Proof. If M is commutative, clearly $\mathbf{TS}(M)$ is also commutative. Now suppose $q_1 F_{\sigma\tau} = p_2$ for each $p_2, q_1 \in Q, \sigma, \tau \in \Sigma$. As M is switching,

$$q_1 F_\sigma = p_1 \Rightarrow p_1 F_\sigma = q_1$$

$$p_1 F_\tau = p_2 \Rightarrow p_2 F_\tau = p_1$$

for each $p_1 \in Q$. Then $p_2 F_{\sigma\tau} = p_2 F_{\tau\sigma} = p_1 F_\sigma = q_1$.
Now assume $qF_{\sigma_1 \cdots \sigma_k} = p \Rightarrow pF_{\sigma_1 \cdots \sigma_k} = q$,
$p, q \in Q, \sigma_i \in \Sigma, (i = 1, \cdots, k)$, and suppose $qF_{\sigma_1 \cdots \sigma_k \sigma_{k+1}} = r, \quad r \in Q$, then

$$pF_{\sigma_{k+1}} = qF_{\sigma_1 \cdots \sigma_k} F_{\sigma_{k+1}} = r$$

from the inductive assumption.
As M is switching, $rF_{\sigma_{k+1}} = p$. Hence

$$\begin{aligned} rF_{\sigma_1 \cdots \sigma_k \sigma_{k+1}} &= rF_{\sigma_1 \cdots \sigma_k} F_{\sigma_{k+1}} = rF_{\sigma_{k+1}} F_{\sigma_1 \cdots \sigma_k} \\ &= pF_{\sigma_1 \cdots \sigma_k} = q. \end{aligned}$$

Therefore $qs = p \Rightarrow ps = q$ for each $p, q \in Q, \quad s \in \mathbf{S}(M)$. □

Indeed, while $M = (Q, \Sigma, F)$ be a switching state machine, $\mathbf{TS}(M)$ is not always a switchboard transformation semigroup.

Proposition 2.1.5. Let $M = (Q, \Sigma, F)$ be a complete switching state machine. Let $e \in S(M)$ be an identity, that is, $e = 1_Q$. Then

$F_{\sigma^2} = F_e$ for each $\sigma \in \Sigma$.

Proof. As M is complete, there exists $p \in Q$ such that $qF_\sigma = p$ for each $q \in Q$, $\sigma \in \Sigma$. As M is switching, $pF_\sigma = q$. Therefore

$$qF_{\sigma^2} = qF_\sigma F_\sigma = (qF_\sigma)F_\sigma = pF_\sigma = q = qF_e,$$

and so $F_{\sigma^2} = F_e$. □

Remark 2.1.6. Let $A = (Q, S)$ be a complete switching transformation semigroup. Then A is a switchboard transformation monoid.

Proof. Let $s \in S$, $s^2 = e$ where e is the identity element of S. Let B be a monoid, such that $bb = e$ for each $b \in B$. Then B is commutative ([2] Exercise 3. 4. (b)). □

Definition 2.1.7. Let $M = (Q, \Sigma, F)$ be a switchboard state machine. If $\sigma \in \Sigma$ exists such that $pF_\sigma = q$ for $p, q \in Q$, we call p and q are in a *switching relation*,
and denote a *switching class* by $[q]_\sigma = \{p, q\}$.

Proposition 2.1.8. Let $M = (Q, \Sigma, F)$ be a switchboard state machine.
Denote $Q_\sigma = \{[q]_\sigma | q \in Q\}$, and $\Sigma_\sigma = \Sigma \backslash \{\sigma\}$ for $\sigma \in \Sigma$.
Put $\alpha = [q]_\sigma, q \in Q$. Define $G : Q_\sigma \times \Sigma_\sigma \to Q_\sigma$ as $\alpha F_\tau^\sigma = [qF_\tau]_\sigma$ for $\alpha \in Q_\sigma, \tau \in \Sigma_\sigma$, then $M_\sigma = (Q_\sigma, \Sigma_\sigma, F^\sigma)$ is a switchboard state machine.

Proof. Let $r \in Q$, then $rF_\sigma \in [r]_\sigma$, and so $[rF_\sigma]_\sigma = [r]_\sigma$.
Let $\alpha, \beta \in Q_\sigma$, put $\alpha = [q]_\sigma, \beta = [p]_\sigma, q, p \in Q$, and suppose $\alpha = \beta$, then $[q]_\sigma = [p]_\sigma$, and so $qF_\sigma = p$ or $q = p$.
Let $\tau, \rho \in \Sigma_\sigma$, suppose $\tau = \rho$. If $q = p$, then
$\alpha F_\tau^\sigma = [qF_\tau]_\sigma = [qF_\rho]_\sigma = [pF_\rho]_\sigma = \beta F_\rho^\sigma$.
If $qF_\sigma = p$, then

$$\begin{aligned} \alpha F_\tau^\sigma &= [qF_\tau]_\sigma = [qF_\rho]_\sigma \\ &= [qF_\rho F_\sigma]_\sigma = [qF_\sigma F_\rho]_\sigma = [pF_\rho]_\sigma \\ &= \beta F_\rho^\sigma. \end{aligned}$$

Because if $rF_\sigma \in [r]_\sigma$ for $\forall r \in Q$, then $[rF_\sigma]_\sigma = [r]_\sigma$. Thus F^σ is well-defined.

Let $\alpha, \beta \in Q_\sigma, \tau \in \Sigma_\sigma$, put $\alpha = [q]_\sigma, \beta = [p]_\sigma$ and suppose $\alpha F^\sigma_\tau = \beta$, then $[qF_\tau]_\sigma = [p]_\sigma$,
that is, $qF_\tau = p$ or $qF_\tau F_\sigma = p$. If $qF_\tau = p$, then $pF_\tau = q$. Hence

$$\beta F^\sigma_\tau = [pF_\tau]_\sigma = [q]_\sigma = \alpha.$$

If $qF_\tau F_\sigma = p$, then

$$qF_\sigma F_\tau = p \quad \Leftrightarrow \quad pF_\tau = qF_\sigma.$$

Hence

$$\beta F^\sigma_\tau = [pF_\tau]_\sigma = [qF_\sigma]_\sigma = [q]_\sigma = \alpha.$$

Thus M_σ is switching.

Let $\alpha \in Q_\sigma, \tau, \rho \in \Sigma_\sigma$, put $\alpha = [q]_\sigma$ for $q \in Q$.
Then

$$\alpha F^\sigma_{\tau\rho} = [qF_{\tau\rho}]_\sigma = [qF_{\rho\tau}]_\sigma = \alpha F^\sigma_{\rho\tau}.$$

Thus M_σ is commutative. □

We call $M_\sigma = (Q_\sigma, \Sigma_\sigma, F^\sigma)$ by a *switchboard state machine for* $\sigma \in \Sigma$.

Theorem 2.1.9. Let $M = (Q, \Sigma, F)$ be a switchboard state machine. Let $M_\sigma = (Q_\sigma, \Sigma_\sigma, F^\sigma)$ be a switchboard state machine for $\sigma \in \Sigma$. Then $M_\sigma \leq M$.

Proof. Define a function $\xi : \Sigma_\sigma \to \Sigma$ by $\xi(\sigma) = \sigma$, $\sigma \in \Sigma$.

Then ξ is clearly a one-one mapping.
Define a partial function $\eta : Q \to Q_\sigma$ by $\eta(q) = [q]_\sigma$, $q \in Q$, then η is clearly surjective.
Let $q \in Q, \tau \in \Sigma_\sigma$, then

$$\eta(q)F^\sigma_\tau = [q]_\sigma F^\sigma_\tau = [qF_\tau]_\sigma = \eta(qF_\tau) = \eta(qF_{\xi(\tau)}).$$

Thus $M_\sigma \leq M$. □

Proposition 2.1.10. Let $M = (Q, \Sigma, F)$, $M' = (Q', \Sigma', F')$ be state machines. Then

$M \times M'$ is a switchboard state machine if and only if both M and M' are switchboard state machines.

Proof. Let $(q, q'), (p, p') \in Q \times Q'$, $(\sigma, \sigma') \in \Sigma \times \Sigma'$. Then

$$\begin{aligned}(F(q, \sigma), F'(q', \sigma')) &= (F \times F')((q, q'), (\sigma, \sigma')) \\ &= (p, p').\end{aligned}$$

Therefore $F(q, \sigma) = p$ and $F'(q', \sigma') = p'$.

As M, M' be switchboard state machines, we have

$$F(q, \sigma) = p \Rightarrow F(p, \sigma) = q, \quad q, p \in Q, \quad \sigma \in \Sigma,$$

and

$$F'(q', \sigma') = p' \Rightarrow F'(p', \sigma') = q', \quad q', p' \in Q', \quad \sigma' \in \Sigma'.$$

Then

$$\begin{aligned}(F \times F')((p, p'), (\sigma, \sigma')) &= (F(p, \sigma), F(p', \sigma')) \\ &= (q, q').\end{aligned}$$

Thus $M \times M'$ is switching.

Let $(q, q') \in Q \times Q'$, $(\sigma, \sigma'), (\tau, \tau') \in \Sigma \times \Sigma'$. Because M and M' are commutative,

$$F(q, \sigma\tau) = F(q, \tau\sigma) \quad \text{and} \quad F'(q', \sigma'\tau') = F'(q', \tau'\sigma').$$

Then

$$\begin{aligned}(F \times F')((q, q'), (\sigma, \sigma')(\tau, \tau')) &= (F \times F')((q, q'), (\sigma\tau, \sigma'\tau')) \\ &= (F(q, \sigma\tau), F'(q', \sigma'\tau')) \\ &= (F(q, \tau\sigma), F'(q', \tau'\sigma')) \\ &= (F \times F')((q, q'), (\tau\sigma, \tau'\sigma')) \\ &= (F \times F')((q, q'), (\tau, \tau')(\sigma, \sigma'))\end{aligned}$$

Thus $M \times M'$ is commutative. Consequently $M \times M'$ is a switchboard state machine.

Conversely, let $F(q, \sigma) = p$, $F'(q', \sigma') = p'$ for each $q, p \in Q$, $q', p' \in Q'$, $\sigma \in \Sigma$, $\sigma' \in \Sigma'$. Then

$$\begin{aligned}(F \times F')((q, q'), (\sigma, \sigma')) &= (F(q, \sigma), F'(q', \sigma')) \\ &= (p, p').\end{aligned}$$

As $M \times M'$ is switching, we have

$$\begin{aligned}(F(p,\sigma), F'(p',\sigma')) &= (F \times F')((p,p'),(\sigma,\sigma')) \\ &= (q,q').\end{aligned}$$

Therefore $F(p,\sigma) = q$, $F'(p',\sigma') = q'$. Thus M and M' are switching. Let $M \times M'$ be commutative. Then clearly M and M' are commutative. So M and M' are switchboard state machines. □

We call the state machine $M = (Q, \{1_Q\}, F)$ by the *trivial machine*. Clearly identity machines are switchboard state machines.

Example 2.1.11. Consider the state machine $M = (Q, \Sigma, F)$ defined by the following table. This is a trivial machine.

F	0	1	2
σ	0	1	2

Proposition 2.1.12. Let $M = (Q, \Sigma, F)$ be a state machine, $M' = (Q', \{1_{Q'}\}, F')$ be a trivial machine , $|\Sigma| \leq |Q'|$, then $M \circ M'$ is switching if and only if M is switching.

Proof. Let $M \circ M'$ be switching, $p, q \in Q$, $q' \in Q'$, $f \in \Sigma^{Q'}$ and suppose $F(q, f(q')) = p$. Then

$$F^\circ((q,q'),(f,1_{Q'})) = (F(q,f(q')), F'(q',1_{Q'})) = (p,q').$$

As $M \circ M'$ is switching, so

$$(F(p,f(q')), F'(q',1_{Q'})) = F^\circ((p,q'),(f,1_{Q'})) = (q,q').$$

Therefore $F(p, f(q')) = q$. As $|\Sigma| \leq |Q'|$, then there exists $f \in \Sigma^{Q'}$ such that $f(Q') = \Sigma$. Thus M is switching.

Conversely let M be switching, $(q,q') \in Q \times Q'$, $r \in Q$, $f \in \Sigma^{Q'}$, and suppose

$$(F(q,f(q')), F'(q',1_{Q'})) = F^\circ((q,q'),(f,1_{Q'})) = (r,q').$$

Hence

$$F(q,f(q')) = r, \quad F'(q',1_{Q'})) = q'.$$

As M is switching, $F(r, f(q')) = q$. Therefore

$$F^{\circ}((r, q'), (f, 1_{Q'})) = (F(r, f(q')), F'(q', 1_{Q'})) = (q, q').$$

Thus $M \circ M'$ is switching. $\square$

2.2. Restricted cascade products

We introduce the restricted cascade product.

Definition 2.2.1. Let $M = (Q, \Sigma, F), M' = (Q', \Sigma', F')$ be complete state machines.
Define the *restricted cascade product* $M \varpi M'$ of M and M' with *respect to* a mapping $\varpi : \Sigma' \to \Sigma$
by

$$M \varpi M' = (Q \times Q', \Sigma', F^{\varpi})$$

where

$$F^{\varpi}((q, q'), \sigma') = (F(q, \varpi(\sigma')), F'(q', \sigma'))$$

for $\sigma' \in \Sigma', (q, q') \in Q \times Q'$.

Example 2.2.2. Consider the state machine $M = (Q, \Sigma, F)$ defined by the following table.

F	0	1
σ	0	0
τ	1	1
ρ	1	0

And consider the state machine $M' = (Q', \Sigma', F')$ defined by the following table.

F'	0	1
σ'	0	0
τ'	1	0

We can hold the state machine $M \varpi M'$ defining $\varpi : \Sigma' \to \Sigma$ by $\varpi(\sigma') = \sigma$, $\varpi(\tau') = \tau$. The partial function F^{ϖ} is presented by the following table.

F^{ϖ}	(0,0)	(0,1)	(1,0)	(1,1)
σ'	(0,0)	(0,0)	(0,0)	(0,0)
τ'	(1,1)	(1,0)	(1,1)	(1,0)

Proposition 2.2.3. Let $M = (Q, \Sigma, F), M' = (Q', \Sigma', F')$ be state machines.
Let $\varpi : \mathbf{S}(M') \to \mathbf{S}(M)$ be a semigroup epimorphism. Then $M\varpi M'$ is a switchboard state machine if and only if both M and M' are switchboard state machines.

Proof. First assume that M and M' are switchboard state machines. Let $(q, q') \in Q \times Q'$, $\sigma', \tau' \in \Sigma'$. Since M and M' are commutative, and ϖ is homomorphism then

$$\begin{aligned} F^{\varpi}((q,q'),\sigma'\tau') &= (F(q,\varpi(\sigma'\tau')), F'(q',\sigma'\tau')) \\ &= (F(q,\varpi(\sigma')\varpi(\tau')), F'(q',\sigma'\tau')) \\ &= (F(q,\varpi(\tau')\varpi(\sigma')), F'(q',\tau'\sigma')) \\ &= (F(q,\varpi(\tau'\sigma')), F'(q',\tau'\sigma')) \\ &= F^{\varpi}((q,q'),\tau'\sigma'). \end{aligned}$$

Thus $M\varpi M'$ is commutative.

Let $(q, q'), (p, p') \in Q \times Q'$, $\sigma' \in \Sigma'$. Suppose

$$F^{\varpi}((q,q'),\sigma') = (F(q,\varpi(\sigma')), F'(q',\sigma')) = (p,p'),$$

then

$$F(q,\varpi(\sigma')) = p, \quad F'(q',\sigma') = p'.$$

As M, M' are switching,

$$F(p,\varpi(\sigma')) = q, \quad F'(p',\sigma') = q'.$$

Then

$$\begin{aligned} F^{\varpi}((p,p'),\sigma') &= (F(p,\varpi(\sigma')), F'(p',\sigma')) \\ &= (q,q'). \end{aligned}$$

Conversely, let $M\varpi M'$ be a switchboard state machine. Let $(q, q') \in Q \times Q', \sigma', \tau' \in \Sigma'$. As $M\varpi M'$ is commutative,

$$\begin{aligned} (F(q,\varpi(\sigma')\varpi(\tau')), F'(q',\sigma'\tau')) &= F^{\varpi}((q,q'),\sigma'\tau') = F^{\varpi}((q,q'),\tau'\sigma') \\ &= (F(q,\varpi(\tau')\varpi(\sigma')), F'(q',\tau'\sigma')) \end{aligned}$$

Therefore

$$F'(q', \sigma'\tau') = F'(q', \tau'\sigma')$$

and

$$F(q, \varpi(\sigma')\varpi(\tau')) = F(q, \varpi(\tau')\varpi(\sigma')) \cdots (*).$$

Thus M' is commutative. And let $\sigma, \tau \in \Sigma$. Because ϖ is surjective, there exists $\sigma'' \in \Sigma'$ such that $\varpi(\sigma'') = \sigma$, and exists $\tau'' \in \Sigma'$ such that $\varpi(\tau'') = \tau$. Then

$$\begin{aligned} F(q, \sigma\tau) &= F(q, \varpi(\sigma'')\varpi(\tau'')) \\ &= F(q, \varpi(\tau'')\varpi(\sigma'')) \\ &= F(q, \tau\sigma) \end{aligned}$$

Thus M is commutative.

Now let $q, p \in Q$, $q', p' \in Q'$, $\sigma' \in \Sigma'$. Suppose

$$F(q, \varpi(\sigma')) = p, \quad F'(q', \sigma') = p'.$$

Then

$$F^{\varpi}((q, q'), \sigma') = (F(q, \varpi(\sigma')), F'(q', \sigma')) = (p, p').$$

As $M\varpi M'$ is switching, we have

$$F^{\varpi}((p, p'), \sigma') = (F(p, \varpi(\sigma')), F'(p', \sigma')) = (q, q'),$$

and so

$$F(p, \varpi(\sigma')) = q, \quad F'(p', \sigma') = q'.$$

Thus M and M' are switching. □

Remark 2.2.4. Let $A = (Q, S), A' = (Q', S')$ be transformation semigroups, where
$S' = \{e'\}$, e' is an identity element of S'. Let Q' be a semigroup such that $\forall q' \in Q'$ are idempotents. And let (S', Q') be a transformation semigroup. Under this condition, define the *cascade product* $A\omega A'$ of A and A' by

$$A\omega A' = (Q \times Q', S')$$

with respect to a homomorphism $\omega : Q' \times S' \to S$ where

$$(q, q')e' = (q\omega(q', e'), q'e')$$

for $\forall(q, q') \in Q \times Q'$, $e' \in S'$. The action of S on $Q \times Q'$ is clearly well-defined.

Let $(q,q') \in Q \times Q'$, $e' \in S'$, then

$$\begin{aligned} ((q,q')e')e' &= ((q\omega(q',e'),q')e' \\ &= (q\omega(q',e')\omega(q',e'),q') \\ &= (q\omega((q',e')(q',e')),q') \\ &= (q\omega(q'q',e'e'),q') \\ &= (q\omega(q',e'),q'e')) \\ &= (q,q')(e'e') \end{aligned}$$

Since $S' = \{e'\}$, the action is faithful, and so $A\omega A'$ is a transformation semigroup.

Consider the following example.

Example 2.2.5. Consider the state machine $M = (Q, \Sigma, F)$, $Q = \{1,2\}$, $\Sigma = \{\sigma\}$, and the action is defined by the following table.

F	1	2
σ	2	1

Consider the state machine $M' = (Q', \Sigma', F')$, $Q' = \{a, b\}$, $\Sigma' = \{1_{Q'}\}$, and the calculation in Q' is defined by the following table. Then it is clear that Q' is an idempotent semigroup.

*	a	b
a	a	a
b	a	b

Construct the cascade product $M\omega M' = (Q \times Q', \Sigma', F^\omega)$ of M and M'. Then there uniquely exists the homomorphism
$\omega : Q' \times \Sigma' \to \Sigma$ defined by

$$\omega(a, 1_{Q'}) = \omega(b, 1_{Q'}) = \sigma.$$

The action is indicated by the following table.

F^ω	(1,a)	(1,b)	(2,a)	(2,b)
$1_{Q'}$	(2,a)	(2,b)	(1,a)	(1,b)

Therefore the action of $\mathbf{TS}(M\omega M')$ is indicated by the following table.

λ	(1,a)	(1,b)	(2,a)	(2,b)
$[1_{Q'}]$	(2,a)	(2,b)	(1,a)	(1,b)
$[(1_{Q'})^2]$	(1,a)	(1,b)	(2,a)	(2,b)

Construct $\mathbf{TS}(M)\omega\mathbf{TS}(M')$. Then there uniquely exists the homomorphism $\omega : Q' \times S' \to S'$ defined by

$$\omega(q', [1_{Q'}]') = \sigma$$

for $\forall(q', [1_{Q'}]') \in Q' \times S'$.

Define $\Phi : \mathbf{S}(M\omega M') \to \mathbf{S}(M)\omega\mathbf{S}(M')$ by

$$\Phi([1_{Q'}]) = \Phi([(1_{Q'})^2]) = [1_{Q'}]'.$$

Φ is uniquely defined. Define $\eta_1 : Q \times Q' \to Q \times Q'$ by $\eta_1((1,a)) = (1,a)$, and let $[1_{Q'}] \in \mathbf{S}(M\omega M')$, $(1,a) \in Q \times Q'$. Then

$$\begin{aligned} \eta_1((1,a))[1_{Q'}] &= (1,a)[1_{Q'}] = (2,a) \\ &\neq (1,a) = \eta_1((1,a)) = \eta_1((1,a)[1_{Q'}]') \end{aligned}$$

Thus η_1 is not a covering function.

Define $\eta_2 : Q \times Q' \to Q \times Q'$ by $\eta_2((1,a)) = (2,a)$, and let $[1_{Q'}] \in \mathbf{S}(M\omega M')$, $(1,a) \in Q \times Q'$. Then

$$\begin{aligned} \eta_2((1,a))[1_{Q'}] &= (2,a)[1_{Q'}] = (1,a) \\ &\neq (2,a) = \eta_2((1,a)) = \eta_2((1,a)[1_{Q'}]') \end{aligned}$$

Thus η_2 is not a covering function.

Define $\eta_3 : Q \times Q' \to Q \times Q'$ by $\eta_3((1,a)) = (1,b)$, and let $[1_{Q'}] \in \mathbf{S}(M\omega M')$, $(1,a) \in Q \times Q'$. Then

$$\begin{aligned} \eta_3((1,a))[1_{Q'}] &= (1,b)[1_{Q'}] = (2,b) \\ &\neq (1,b) = \eta_3((1,a)) = \eta_3((1,a)[1_{Q'}]') \end{aligned}$$

Thus η_3 is not a covering function.

Define $\eta_4 : Q \times Q' \to Q \times Q'$ by $\eta_4((1,a)) = (2,b)$, and let $[1_{Q'}] \in \mathbf{S}(M\omega M')$, $(1,a) \in Q \times Q'$. Then

$$\begin{aligned} \eta_4((1,a))[1_{Q'}] &= (2,b)[1_{Q'}] = (1,b) \\ &\neq (2,b) = \eta_4((1,a)) = \eta_4((1,a)[1_{Q'}]') \end{aligned}$$

Thus η_4 is not a covering function.

Consequently $\mathbf{TS}(M)\omega\mathbf{TS}(M') \leq \mathbf{TS}(M\omega M')$ does not hold.

Lemma 2.2.6. Let $M = (Q, \Sigma, F)$ be a complete switchboard state machine. And let $e = \{1_Q, \sigma^{2b}\}$, $\sigma \in \Sigma$, $b \in N$, where $e \in \mathbf{S}(M)$ is the identity element. Then the decomposition of $s \in \mathbf{S}(M)$ is distinctly unique.

Proof. We can denote $s = [\sigma_1\sigma_2 \cdots \sigma_i \cdots \sigma_j \cdots \sigma_n]$ for $s \in \mathbf{S}(M)$, $\sigma_k \in \Sigma, k = 1, \cdots, n$. If $\sigma_i = \sigma_j, i \neq j$, then

$$\begin{aligned} s = [\sigma_1\sigma_2 \cdots \sigma_i \cdots \sigma_j \cdots \sigma_n] &= [\sigma_1\sigma_2 \cdots (\sigma_i\sigma_j)\sigma_{i+1} \cdots \sigma_{j-1}\sigma_{j+1} \cdots \sigma_n] \\ &= [\sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_{i+1} \cdots \sigma_{j-1}\sigma_{j+1} \cdots \sigma_n] \end{aligned}$$

by the commutativity of M and from Proposition 3.5. Therefore we can denote distinctly $s = [\sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_{i+1} \cdots \sigma_n]$ if the number of $\sigma_i \in s$ is even, and $s = [\sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_i\sigma_{i+1} \cdots \sigma_n]$ if the number of $\sigma_i \in s$ is odd. Now suppose $s = [\sigma_1\sigma_2 \cdots \sigma_n] = [\tau_1\tau_2 \cdots \tau_m], \sigma_i, \tau_j \in \Sigma$ distinctly where $m \neq n$ or $\exists \sigma_k, \tau_l \in \Sigma$ such that $\sigma_k \neq \tau_l$. Then

$$qF_{\sigma_1}F_{\sigma_2 \cdots \sigma_n} = qF_{\tau_1}F_{\tau_2 \cdots \tau_m},$$

for each $q \in Q$. As M is switching, $qF_{\tau_1}F_{\tau_2 \cdots \tau_m}F_{\sigma_2 \cdots \sigma_n} = qF_{\sigma_1}$, for each $q \in Q$. That is, $[\sigma_1] = [\tau_1 \cdots \tau_m\sigma_2 \cdots \sigma_n]$. Hence

$$[e] = [\sigma_1][\sigma_1] = [\sigma_1][\tau_1 \cdots \tau_m\sigma_2 \cdots \sigma_n] = [\sigma_1\tau_1 \cdots \tau_m\sigma_2 \cdots \sigma_n].$$

where $\sigma_1\tau_1 \cdots \tau_m\sigma_2 \cdots \sigma_n \neq e$. This contradicts with the condition. Thus the assertion was verified. □

Proposition 2.2.7. Let $M = (Q, \Sigma, F)$ be a state machine, and $M' = (Q', \Sigma', F')$ be a complete switchboard state machine such that $e' = \{1_{Q'}, (\sigma')^{2b}\} \in \mathbf{S}(M')$ $\sigma' \in \Sigma'$, $b \in N$.
And let $\varpi : \mathbf{S}(M') \to \mathbf{S}(M)$ be a semigroup homomorhism. Then $\mathbf{TS}(M)\varpi\mathbf{TS}(M')$ is a transformation semigroup.

Proof. Define $\varpi : \mathbf{S}(M') \to \mathbf{S}(M)$ by

$$\varpi([\alpha']') = [\varpi(\sigma_1') \cdots \varpi(\sigma_n')] \quad \ldots (\star),$$

where $\forall[\alpha']' \in \mathbf{S}(M'), \sigma'_i \in \Sigma', \alpha' = \sigma'_1 \cdots \sigma'_n$.
Let $[\alpha'_1]', [\alpha'_2]' \in \mathbf{S}(M'), \alpha'_1 = \sigma'_1 \cdots \sigma'_n, \alpha'_2 = \tau'_1 \cdots \tau'_m, \sigma'_i, \tau'_j \in \Sigma'$ and suppose $[\alpha'_1]' = [\alpha'_2]'$. Then since $\sigma'_i = \tau'_i, m = n$ from Lemma 4.6,

$$\varpi([\alpha'_1]') = [\varpi(\sigma'_1) \cdots \varpi(\sigma'_n)] = [\varpi(\tau'_1) \cdots \varpi(\tau'_m)] = \varpi([\alpha'_2]').$$

Thus ϖ is well-defined.

Define an action of $\mathbf{S}(M')$ on $Q \times Q'$ by

$$(q, q')s' = (q\varpi(s'), q's')$$

for $(q, q') \in Q \times Q', s' \in \mathbf{S}(M')$.
Let $(q_1, q'_1), (q_2, q'_2) \in Q \times Q'$, and suppose $(q_1, q'_1) = (q_2, q'_2)$.
Then $q_1 = q_2$ and $q'_1 = q'_2$. Let $s'_1, s'_2 \in \mathbf{S}(M')$, and suppose $s'_1 = s'_2$.
Then

$$(q_1, q'_1)s'_1 = (q_1\varpi(s'_1), q'_1s'_1) = (q_2\varpi(s'_2), q'_2s'_2) = (q_2, q'_2)s'_2.$$

Hence the action is well-defined.

Let $(q, q') \in Q \times Q', s'_1, s'_2 \in \mathbf{S}(M')$. Then

$$\begin{aligned}
((q, q')s'_1)s'_2 &= (q\varpi(s'_1), q's'_1)s'_2 \\
&= ((q\varpi(s'_1))\varpi(s'_2), (q's'_1)s'_2) \\
&= (q(\varpi(s'_1)\varpi(s'_2)), q'(s'_1s'_2)) \\
&= (q\varpi(s'_1s'_2), q'(s'_1s'_2)) \\
&= (q, q')(s'_1s'_2).
\end{aligned}$$

And

$$\begin{aligned}
(q, q')s'_1 = (q, q')s'_2 &\Leftrightarrow (q\varpi(s'_1), q's'_1) = (q\varpi(s'_2), q's'_2) \\
&\Leftrightarrow q\varpi(s'_1) = q\varpi(s'_2), q's'_1 = q's'_2 \\
&\Rightarrow s'_1 = s'_2.
\end{aligned}$$

Hence the action is faithful. □

Theorem 2.2.8. Let $M = (Q, \Sigma, F)$ be a state machine, and $M' = (Q', \Sigma', F')$ a complete switchboard state machine such that $e' = \{1_{Q'}, (\sigma')^{2b}\}, \sigma' \in \Sigma', b \in N, e' \in \mathbf{S}(M')$ and let $\varpi : \mathbf{S}(M') \to \mathbf{S}(M)$ be a homomorphism. Then

$$\mathbf{TS}(M\varpi M') \cong \mathbf{TS}(M)\varpi\mathbf{TS}(M').$$

Proof. Define $\varphi : \mathbf{S}(M\varpi M') \to \mathbf{S}(M')$ by

$$\varphi([\alpha']_\varpi) = [\alpha']'$$

where $[\alpha']_\varpi \in \mathbf{S}(M\varpi M'), [\alpha']' \in \mathbf{S}(M'), \alpha' \in (\Sigma')^+$.
Let $[\alpha']_\varpi, [\beta']_\varpi \in \mathbf{S}(M\varpi M')$ and suppose $[\alpha']_\varpi = [\beta']_\varpi$. Then

$$(q, q')F^\varpi_{\alpha'} = (q, q')F^\varpi_{\beta'}$$

for all $(q, q') \in Q \times Q'$. That is,

$$(F(q, \varpi(\alpha')), F'(q', \alpha')) = (F(q, \varpi(\beta')), F'(q', \beta')).$$

Thus

$$F(q, \varpi(\alpha')) = F(q, \varpi(\beta'))$$

and

$$F'(q', \alpha') = F'(q', \beta').$$

That is,

$$F_{\varpi(\alpha')} = F_{\varpi(\beta')}, F'_{\alpha'} = F'_{\beta'}.$$

Then $[\alpha']' = [\beta']'$. Therefore φ is well-defined and clearly bijective.
Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q\times Q'}$. Then

$$\begin{aligned}
\eta((q, q')[\alpha']_\varpi) &= (q, q')[\alpha']_\varpi = (q, q')F^\varpi_{\alpha'} \\
&= (F(q, \varpi(\alpha')), F'(q', \alpha')) \\
&= (q[\varpi(\alpha')], q'[\alpha']') \\
&= (q\varpi([\alpha']'), q'[\alpha']') \qquad \text{(from Proposition 2.2.7. } (\star)) \\
&= \eta((q, q'))\varphi([\alpha']_\varpi)
\end{aligned}$$

for $\forall[\alpha']_\varpi \in \mathbf{S}(M\varpi M'), \forall(q, q') \in Q \times Q'$.
Thus the assertion was proved. □

Corollary 2.2.9. Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines such that $\Sigma = \Sigma'$. And let other conditions be the same as Theorem 4.8. Then

$$\mathbf{TS}(M) \wedge \mathbf{TS}(M') \cong \mathbf{TS}(M)\varpi\mathbf{TS}(M')$$

for suitable epimorphisms $\theta : \Sigma^+ \to \mathbf{S}(M), \theta' : \Sigma^+ \to \mathbf{S}(M')$.

Proof. Clearly there exists a semigroup isomorphism $\varpi : \mathbf{S}(M') \to \mathbf{S}(M)$ and $M \wedge M' \cong M\varpi M'$. Thus the assertion holds from Theorem 1.3.12 and 2.2.8. □

Proposition 2.2.10. Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then there exists $\omega : Q' \times \Sigma' \to \Sigma$ for all $\varpi : \Sigma' \to \Sigma$ such that

$$M\varpi M' \cong M\omega M'.$$

Proof. Define $\omega : Q' \times \Sigma' \to \Sigma$ by

$$\omega(q', \sigma') = \varpi(p_2(q', \sigma')) \quad \text{for} \quad (q', \sigma') \in Q' \times \Sigma'$$

where $p_2 : Q' \times \Sigma' \to \Sigma'$ is a projection mapping. This definition is clearly well-defined.

Let $\xi : \Sigma' \to \Sigma'$ be $1_{\Sigma'}$, and $\eta : Q \times Q' \to Q \times Q'$ $1_{Q\times Q'}$. Then

$$\begin{aligned} \eta((q,q')F^{\varpi}_{\sigma'}) &= (q,q')F^{\varpi}_{\sigma'} \\ &= (F(q,\varpi(\sigma')), F'(q',\sigma')) \\ &= (F(q,\omega(q',\sigma')), F'(q',\sigma')) \\ &= (q,q')F^{\omega}_{\sigma'} \\ &= (\eta(q,q'))F^{\omega}_{\xi(\sigma')}. \end{aligned}$$

Thus $M\varpi M' \cong M\omega M'$. □

Corollary 2.2.11. Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. And let other conditions be the same as Theorem 2.2.8. Then

$$\mathbf{TS}(M)\varpi\mathbf{TS}(M') \leq \mathbf{TS}(M) \circ \mathbf{TS}(M').$$

Proof. From Theorem 1.4.4, 2.2.8, and Proposition 2.2.10,

$$\begin{aligned} \mathbf{TS}(M)\varpi\mathbf{TS}(M') &\cong \mathbf{TS}(M\varpi M') \\ &\cong \mathbf{TS}(M\omega M') \\ &\leq \mathbf{TS}(M) \circ \mathbf{TS}(M'). \end{aligned}$$

Theorem 2.2.12. Let $A = (Q, S)$, $A' = (Q', S')$, $B = (P, T)$ be transformation semigroups such that $A \leq B$. Then there exists a function $\varpi_2 : S' \to T$ for each function $\varpi_1 : S' \to S$ such that

$$A\varpi_1 A' \leq B\varpi_2 A'.$$

Proof. Suppose that $\phi : P \to Q$ is a surjective partial covering function, and given $s \in S$ there exists an element $t_s \in T$ covering s. Define $\phi : P \times Q' \to Q \times Q'$ by

$$\Phi((p, q')) = (\phi(p), q')$$

for $(p, q') \in P \times Q'$. ϕ is clearly surjective partial function.

Now let $s' \in S'$, define $\varpi_2 : S' \to T$ by

$$\varpi_2(s') = t_{\varpi_1(s')}.$$

ϖ_2 is clearly well-defined. Let $p \in P$, $s \in S$, then $\phi(p)s \subseteq \phi(pt_s)$. Therefore $\phi(p)\varpi_1(s') \subseteq \phi(pt_{\varpi_1(s')})$ for $\forall s' \in S'$.

Let $(p, q') \in P \times Q'$, $s' \in S'$, then

$$\begin{aligned}
\Phi((p, q'))s' &= (\phi(p), q')s' \\
&= (\phi(p)\varpi_1(s'), q') \\
&\subseteq (\phi(pt_{\varpi_1(s')}), q') \\
&= (\phi(p\varpi_2(s')), q') \\
&= \Phi(p\varpi_2(s'), q') \\
&= \Phi((p, q')s'). \qquad \square
\end{aligned}$$

2.3. Cartesian compositions

Now we study the cartesian compositions of state machines.

Definition 2.3.1.([7]) Let $M = (Q, \Sigma, F)$, $M' = (Q', \Sigma', F')$ be complete state machines. Define the *cartesian composition* $M \otimes M'$ of M and M' by

$$M \otimes M' = (Q \times Q', \Sigma \cup \Sigma', G)$$

where

$$G((q,q'),\sigma) = (F(q,\sigma),q'), \quad \sigma \in \Sigma$$

$$G((q,q'),\sigma') = (q,F(q',\sigma')), \quad \sigma' \in \Sigma'$$

for $(q,q') \in Q \times Q'$.

Example 2.3.2. Consider the state machine $M = (Q,\Sigma,F)$ defined by the following table.

F	1	2
σ	2	1
τ	1	2

Consider the state machine $M' = (Q',\Sigma',F')$ defined by the following table.

F	1	2
σ'	2	1

Then the cartesian composition $M \otimes M'$ is presented by the following table.

G	(1,1)	(1,2)	(2,1)	(2,2)
σ	(2,1)	(2,2)	(1,1)	(1,2)
τ	(1,1)	(1,2)	(2,1)	(2,2)
σ'	(1,2)	(1,1)	(2,2)	(2,1)

Definition 2.3.3. Define $G : (Q \times Q') \times \Sigma^+ \to Q \times Q'$ inductively by

$$G((q,q'),\sigma\alpha) = G((qF_\sigma,q'),\alpha)$$

where $(q,q') \in Q \times Q', \sigma \in \Sigma, \alpha \in \Sigma^+$.
Similarly define $G : (Q \times Q') \times (\Sigma')^+ \to Q \times Q'$ by

$$G((q,q'),\sigma'\alpha') = G((q,q'F'_\sigma),\alpha')$$

where $\sigma' \in \Sigma'$ and $\alpha' \in (\Sigma')^+$.

Proposition 2.3.4. Let $M = (Q,\Sigma,F)$ and $M' = (Q',\Sigma',F')$ be state machines. Then $M \otimes M'$ is a switchboard state machine if and only if both M and M' are switchboard state machines.

Proof. Let $(q, q') \in Q \times Q'$, $\sigma, \tau \in \Sigma$. As M is commutative. Then

$$\begin{aligned} G((q,q'),\sigma\tau) &= G((qF_\sigma, q'), \tau) = (qF_\sigma F_\tau, q') = (qF_{\sigma\tau}, q') \\ &= (qF_{\tau\sigma}, q') = (qF_\tau F_\sigma, q') = G((qF_\tau, q'), \sigma) \\ &= G((q,q'), \tau\sigma). \end{aligned}$$

Let $(q, q') \in Q \times Q'$, $\sigma', \tau' \in \Sigma'$. As M' is commutative, similarly

$$G((q,q'),\sigma'\tau') = G((q,q'),\tau'\sigma').$$

Let $\sigma \in \Sigma, \tau' \in \Sigma'$. Then

$$\begin{aligned} G((q,q'),\sigma\tau') &= G((qF_\sigma, q'), \tau') = (qF_\sigma, q'F'_{\tau'}) \\ &= G((q, q'F'_{\tau'}), \sigma) = G((q,q'), \tau'\sigma). \end{aligned}$$

Similarly, let $\sigma' \in \Sigma'$, $\tau \in \Sigma$. Then

$$G((q,q'),\sigma'\tau) = G((q,q'),\tau\sigma').$$

Thus $M \otimes M'$ is commutative.

Let $(q, q'), (p, p') \in Q \times Q', \sigma \in \Sigma$, and suppose

$$(F(q,\sigma), q') = G((q,q'),\sigma) = (p, p'),$$

then

$$F(q,\sigma) = p \quad \text{and} \quad q' = p'.$$

As M is switching, $F(p, \sigma) = q$. Then

$$G((p,p'),\sigma) = (F(p,\sigma), p') = (q, q').$$

Similarly, let $\sigma' \in \Sigma'$, and suppose $G((q, q'), \sigma') = (p, p')$. Then

$$G((p,p'),\sigma') = (q,q').$$

Thus $M \otimes M'$ is switching.

Conversely, let $q \in Q$, $\sigma, \tau \in \Sigma$, $q' \in Q'$. As $M \otimes M'$ is commutative,

$$\begin{aligned} (qF_{\sigma\tau}, q') &= G((qF_\sigma, q'), \tau) \\ &= G((q,q'), \sigma\tau) \\ &= G((qF_\sigma, q'), \tau\sigma) \\ &= G((qF_\tau, q'), \sigma) \\ &= (qF_{\tau\sigma}, q'). \end{aligned}$$

So $qF_{\tau\sigma} = qF_{\sigma\tau}$. Thus M is commutative. Similarly it can be proved that M' is commutative.

And now let $q \in Q$, $\sigma, \tau \in \Sigma$, $q' \in Q'$. Suppose $F(p, \sigma) = q$. Then

$$G((p, q'), \sigma) = (F(p, \sigma), q') = (q, q').$$

As $M \otimes M'$ is switching,

$$(F(q, \sigma), q') = G((q, q'), \sigma) = (p, q').$$

So $F(q, \sigma) = p$. Thus M is switching. Similarly it can be proved that M' is switching.

Consequently M and M' are switchboard state machines. □

Remark 2.3.5. Consider the state machines $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ defined by the same as Example 2.3.2. Then the direct product $M \times M'$ is presented by the following table.

$F \times F'$	(1,1)	(1,2)	(2,1)	(2,2)
(σ, σ')	(2,2)	(2,1)	(1,2)	(1,1)
(τ, σ')	(1,2)	(1,1)	(2,2)	(2,1)

Now consider the relation between $M \otimes M'$ and $M \times M'$.

Let $\xi : \Sigma \cup \Sigma' \to \Sigma \times \Sigma'$ be any function and let $\forall (q, q') \in Q \times Q'$, $\forall \tau \in \Sigma \cup \Sigma'$. Then

$$\eta((q, q'))G_\tau = \eta((q, q')).$$

And let

$$\eta((q, q')(F \times F')_{\xi(\tau)}) = \eta((q_1, q_1')).$$

Then clearly $(q, q') \neq (q_1, q_1')$. As η is bijective, so $\eta((q, q')) \neq \eta((q_1, q_1'))$. Thus $M \otimes M' \leq M \times M'$ does not hold.

Theorem 2.3.6. Let $M = (Q, \Sigma, F), M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M \otimes M') \cong \mathbf{TS}(M)^1 \times \mathbf{TS}(M')^1$$

Proof. Let $[x]_\otimes \in \mathbf{S}(M \otimes M')$, $x \in (\Sigma \cup \Sigma')$. Then by the proof of Proposition 5.4, we can indicate $[x]_\otimes = [\alpha\beta]_\otimes$, $\alpha \in \Sigma^*, \beta \in (\Sigma')^*$.

Define $g : \mathbf{S}(M \otimes M') \to \mathbf{S}(M)^1 \times \mathbf{S}(M')^1$ by

$$g([\alpha\beta]_\otimes) = ([\alpha], [\beta]')$$

for $[\alpha\beta]_\otimes \in \mathbf{S}(M \otimes M')$, $[\alpha] \in \mathbf{S}(M)^1, [\beta]' \in \mathbf{S}(M')^1$.

Let $[\alpha_1\beta_1]_\otimes, [\alpha_2\beta_2]_\otimes \in \mathbf{S}(M \otimes M')$ and $\forall (q, q') \in Q \times Q'$. Then

$$\begin{aligned}
[\alpha_1\beta_1]_\otimes = [\alpha_2\beta_2]_\otimes &\Leftrightarrow G((q,q'), \alpha_1\beta_1) = G((q,q'), \alpha_2\beta_2) \quad \text{for} \\
&\Leftrightarrow (qF_{\alpha_1}, q'F'_{\beta_1}) = (qF_{\alpha_2}, q'F'_{\beta_2}) \\
&\Leftrightarrow qF_{\alpha_1} = F_{\alpha_2}, \quad F'_{\beta_1} = F'_{\beta_2} \\
&\Leftrightarrow [\alpha_1] = [\alpha_2], \quad [\beta_1]' = [\beta_2]' \\
&\Leftrightarrow ([\alpha_1], [\beta_1]') = ([\alpha_2], [\beta_2]')
\end{aligned}$$

Hence g is well-defined, and clearly bijective.

Define $\eta : Q \times Q' \to Q \times Q'$ by $\eta = 1_{Q \times Q'}$.

$$\begin{aligned}
\eta((q,q')[\alpha\beta]_\otimes) &= (q,q')[\alpha\beta]_\otimes \\
&= (q\alpha, q'\beta) \\
&= (q[\alpha], q'[\beta]') \\
&= (q,q')([\alpha], [\beta]') \\
&= \eta((q,q'))g([\alpha], [\beta]')
\end{aligned}$$

for $\forall [\alpha\beta]_\otimes \in \mathbf{S}(M \otimes M'), \forall (q,q') \in Q \times Q'$. Thus $\mathbf{TS}(M \otimes M') \cong \mathbf{TS}(M)^1 \times \mathbf{TS}(M')^1$. □

Theorem 2.3.7. Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M) \times \mathbf{TS}(M') \leq \mathbf{TS}(M \otimes M')$$

Proof. Define $g : \mathbf{S}(M) \times \mathbf{S}(M') \to \mathbf{S}(M \otimes M')$ by

$$g([\alpha], [\beta]') = [\alpha\beta]_\otimes.$$

Let $([\alpha_1], [\beta_1]'), ([\alpha_2], [\beta_2]') \in \mathbf{S}(M) \times \mathbf{S}(M')$ and $\forall (q,q') \in Q \times Q'$. Then

$$([\alpha_1], [\beta_1]') = ([\alpha_2], [\beta_2]') \;\Rightarrow\; [\alpha_1] = [\alpha_2], \quad [\beta_1]' = [\beta_2]'$$

$$\Rightarrow \quad qF_{\alpha_1} = F_{\alpha_2}, \quad F'_{\beta_1} = F'_{\beta_2}$$
$$\Rightarrow \quad (qF_{\alpha_1}, q'F'_{\beta_1}) = (qF_{\alpha_2}, q'F'_{\beta_2})$$
$$\Rightarrow \quad G((q,q'),\alpha_1\beta_1) = G((q,q'),\alpha_2\beta_2) \quad \text{for}$$
$$\Rightarrow \quad [\alpha_1\beta_1]_\otimes = [\alpha_2\beta_2]_\otimes.$$

Thus g is well-defined. Let $(q,q') \in Q \times Q'$,
$([\alpha],[\beta]') \in \mathbf{S}(M) \times \mathbf{S}(M')$. Then, as in the proof of Theorem 2.3.6, we have

$$\eta((q,q')([\alpha],[\beta]')) \quad = \quad \eta((q,q'))[\alpha\beta]_\otimes.$$

Therefore $\mathbf{TS}(M) \times \mathbf{TS}(M') \leq \mathbf{TS}(M \otimes M')$. □

From Theorem 1.4.1 and 2.3.7, the following assertion holds.

Corollary 2.3.8. Let $M = (Q,\Sigma,F)$ and $M' = (Q',\Sigma',F')$ be state machines. Then

$$\mathbf{TS}(M \times M') \leq \mathbf{TS}(M \otimes M').$$

Definition 2.3.9. Let $A = (Q,S)$ and $A' = (Q',S')$ be transformation semigroups. Let
$ss' = s's$ for $\forall s \in S$, $\forall s' \in S'$. Define the *quasi-direct product* $A \otimes A'$ of A and A' by

$$A \otimes A' = (Q \times Q', (S \cup S')^+)$$

where $(S \cup S')^+$ is a semigroup generated from all concatenations of $s \in S$ and $s' \in S'$. The action is given by

$$(q,q')s = (qs,q') \quad \text{for} \quad s \in S$$

and

$$(q,q')s' = (q,q's') \quad \text{for} \quad s' \in S'.$$

Proposition 2.3.10. Let $A = (Q,S)$ and $A' = (Q',S')$ be transformation semigroups satisfying the Definition 2.3.9, then $A \otimes A'$ is a transformation semigroup.

Proof. Let $\alpha_1, \alpha_2 \in (S \cup S')^+$. We can indicate

$$\alpha_1 = s_1 s_1', \quad \alpha_2 = s_2 s_2'$$

for $s_1, s_2 \in S \cup \{\Lambda\}$, $s_1', s_2' \in S' \cup \{\Lambda\}$. Let $(q, q') \in Q \times Q'$. Then

$$\begin{aligned}
((q, q')\alpha_1)\alpha_2 &= ((q, q')s_1 s_1')s_2 s_2' \\
&= (q s_1, q' s_1')s_2 s_2' \\
&= ((q s_1)s_2, (q' s_1')s_2') \\
&= (q(s_1 s_2), q'(s_1' s_2')) \\
&= (q, q')(s_1 s_2)(s_1' s_2') \\
&= (q, q')(s_1 s_1')(s_2 s_2') \\
&= (q, q')(\alpha_1 \alpha_2)
\end{aligned}$$

Suppose $(q, q')\alpha_1 = (q, q')\alpha_2$. Then

$$\begin{aligned}
(q s_1, q' s_1') = (q s_2, q' s_2') &\Leftrightarrow q s_1 = q s_2, \quad q' s_1' = q' s_2' \\
&\Rightarrow s_1 = s_2, \quad s_1' = s_2' \\
&\Rightarrow \alpha_1 = s_1 s_1' = s_2 s_2' = \alpha_2
\end{aligned}$$

Thus the action of $(S \cup S')^+$ on $Q \times Q'$ is faithful. □

Theorem 2.3.11. Let $M = (Q, \Sigma, F)$ and $M' = (Q', \Sigma', F')$ be state machines. Then

$$\mathbf{TS}(M \otimes M') \cong \mathbf{TS}(M) \otimes \mathbf{TS}(M')$$

Proof. Define $g : \mathbf{S}(M \otimes M') \to \mathbf{S}(M) \otimes \mathbf{S}(M')$ by

$$g([\alpha\beta]_\otimes) = [\alpha][\beta]'$$

where $[\alpha\beta]_\otimes \in \mathbf{S}(M \otimes M')$, $[\alpha][\beta]' \in \mathbf{S}(M) \otimes \mathbf{S}(M')$, $\alpha \in \Sigma^*$, $\beta \in (\Sigma')^*$. Then g is clearly well-defined and bijective.

Let $[\alpha_1\beta_1]_\otimes, [\alpha_2\beta_2]_\otimes \in \mathbf{S}(M \otimes M')$. Then

$$\begin{aligned}
g([\alpha_1\beta_1]_\otimes[\alpha_2\beta_2]_\otimes) &= g([\alpha_1\beta_1\alpha_2\beta_2]_\otimes) \\
&= g([\alpha_1\alpha_2\beta_1\beta_2]_\otimes) \\
&= [\alpha_1\alpha_2][\beta_1\beta_2]' \\
&= [\alpha_1][\alpha_2][\beta_1]'[\beta_2]' \\
&= [\alpha_1][\beta_1]'[\alpha_2][\beta_2]' \\
&= g([\alpha_1\beta_1]_\otimes)g([\alpha_2\beta_2]_\otimes)
\end{aligned}$$

Hence g is a semigroup homomorphism.

Define $f : Q \times Q' \to Q \times Q'$ by $f = 1_{Q \times Q'}$. Then

$$\begin{aligned} f((q, q')[\alpha\beta]_{\otimes}) &= (q, q')[\alpha\beta]_{\otimes} \\ &= (q\alpha, q'\beta) \\ &= (q[\alpha], q'[\beta]') \\ &= (q, q')([\alpha][\beta]') \\ &= f((q, q'))g([\alpha\beta]_{\otimes}) \end{aligned}$$

for $\forall (q, q') \in Q \times Q'$, $[\alpha\beta]_{\otimes} \in \mathbf{S}(M \otimes M')$. Thus (f, g) is a state machine isomorphism. □

3. Fuzzy finite switchboard state machines

3.1. Introduction

The concept of fuzzy sets was introduced by Zadeh [8],[9] for the first time. The algebraic techniques of automata theory were used in Holcombe [1]. The concept of fuzzy finite state machines was given by Malik, Mordeson and Sen [10]. In this paper, we introduce the concept of fuzzy finite switchboard state machines and fuzzy transformation semigroups, and examine relations among those systems.

A fuzzy finite state machine (ffsm) is a triple $M = (Q, X, \mu)$ where Q and X are finite nonempty sets and μ is a fuzzy subset of $Q \times X \times Q$, i.e., $\mu : Q \times X \times Q \to [0, 1]$. Let X^* denote the set of all words of elements of X of finite length. Let λ be the empty word in X^* and $|x|$ be the length of $x \in X^*$. Let $Im(\mu)$ be the image of the fuzzy subset μ.

$\mu^* : Q \times X^* \times Q \to [0, 1]$ is defined by

$$\mu^*(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

and

$$\mu^*(q, xa, p) = \vee\{\mu^*(q, x, r) \wedge \mu(r, a, p) | r \in Q\}$$

$\forall x \in X^*$, $\forall a \in X$ and $\forall q, p \in Q$.

Examples

(1) Most simple cases are where Q and X are both singletons. Let $Q = \{0\}$ and $X = \{x\}$. Then we can have $\mu : Q \times X \times Q \to [0, 1]$ defined by $\mu(0, x, 0) = 0.4$.

Q	X	Q	μ
0	x	0	0.4

(2) Suppose that $|Q| = 1$ and X is any finite set, we can't get very different examples. Let $Q = \{0\}$ and $X = \{x, y\}$. Then we can have $\mu : Q \times X \times Q \to [0, 1]$ defined by $\mu(0, x, 0) = 0.7$ and $\mu(0, y, 0) = 0.3$.

Q	X	Q	μ
0	x	0	0.7
0	y	0	0.3

Lemma 3.1.1.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Then

$$\mu^*(q, xy, p) = \vee\{\mu^*(q, x, r) \wedge \mu(r, y, p) | r \in Q\}$$

$\forall x, y \in X^*$ and $\forall q, p \in Q$.

Proof. Let $q, p \in Q$ and $x, y \in X^*$. We prove the result by induction on $|y| = n$. If $n = 0$, then $y = \lambda$. Hence

$$xy = x\lambda = x.$$

Therefore

$$\begin{aligned} \mu^*(q, xy, p) = \mu^*(q, x, p) &= \vee\{\mu^*(q, x, r) \wedge \mu^*(r, \lambda, p) | r \in Q\} \\ &= \vee\{\mu^*(q, x, r) \wedge \mu^*(r, y, p) | r \in Q\}. \end{aligned}$$

Thus the result is true for $n = 0$. Suppose now the result is true for $\forall u \in X^*$ such that $|u| = n - 1$, $n > 0$. Let $y = ua$ where $a \in X$ and $u \in X^*$ and $|u| = n - 1$, $n > 0$. Then

$$\begin{aligned} \mu^*(q, xy, p) &= \mu^*(q, xua, p) \\ &= \vee\{\mu^*(q, xu, r) \wedge \mu(r, a, p) | r \in Q\} \\ &= \vee\{(\vee\{\mu^*(q, x, s) \wedge \mu^*(s, u, r) | s \in Q\}) \wedge \mu(r, a, p) | r \in Q\} \\ &= \vee\{(\vee\{\mu^*(q, x, s) \wedge \mu^*(s, u, r) \wedge \mu(r, a, p) | r, s \in Q\} \\ &= \vee\{\mu^*(q, x, s) \wedge (\vee\{\mu^*(s, u, r) \wedge \mu(r, a, p) | r \in Q\}) | s \in Q\} \\ &= \vee\{\mu^*(q, x, s) \wedge \mu^*(s, ua, p) | s \in Q\} \\ &= \vee\{\mu^*(q, x, s) \wedge \mu^*(s, y, p) | s \in Q\}. \end{aligned}$$

The result now follows by induction. □

Definition 3.1.2.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Define a relation $\equiv$ on X^* by

$$x \equiv y, \forall x, y \in X^* \Leftrightarrow \mu^*(q, x, p) = \mu^*(q, y, p), \forall q, p \in Q.$$

Theorem 3.1.3.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Then $\equiv$ is a congruence relation on X^*.

Proof. Clearly, $\equiv$ is an equivalence relation on X^*. Let $x \equiv y$ for $x, y \in X^*$. And let $z \in X^*$ and $p, q \in Q$. Then

$$\begin{aligned} \mu^*(q, xz, p) &= \vee\{\mu^*(q, x, r) \wedge \mu^*(r, z, p) | r \in Q\} \\ &= \vee\{\mu^*(q, y, r) \wedge \mu^*(r, z, p) | r \in Q\} \\ &= \mu^*(q, yz, p). \end{aligned}$$

Therefore $xz \equiv yz$. Similarly, $zx \equiv zy$. Thus $\equiv$ is a congruence relation on the semigroup X^*. □

Let $x \in X^*$. Define $[x]$ by $[x] = \{y \in X^* | x \equiv y\}$ and define $E(M) = \{[x] | x \in X^*\}$.

Theorem 3.1.4.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Define a binary operation $*$ on $E(M)$ by $\forall [x], [y] \in E(M), [x] * [y] = [xy]$. Then $(E(M), *)$ is a finite monoid.

Proof. Let $[x_1], [x_2], [y_1], [y_2] \in E(M)$ such that $[x_1] = [x_2], [y_1] = [y_2]$. Then

$$\begin{aligned} [x_1] * [y_1] &= [x_1 y_1] = [x_2 y_1] \\ &= [x_2 y_2] = [x_2] * [y_2]. \end{aligned}$$

Therefore $*$ is well-defined.

Let $[x], [y], [z] \in E(M)$. Then

$$\begin{aligned} [x] * ([y] * [z]) &= [x] * [yz] = [x(yz)] \\ &= [(xy)z] = [xy] * [z] \\ &= ([x] * [y]) * [z]. \end{aligned}$$

Therefore $*$ is associative.

Now let $\forall [x] \in E(M)$. Then $[x] * [\lambda] = [x\lambda] = [x] = [\lambda x] = [\lambda] * [x]$ Therefore $[\lambda]$ is the identity of $(E(M), *)$. Hence $(E(M), *)$ is a monoid.

Let $x \in X^*$ and $x = x_1 x_2 \cdots x_n$ where $x_1, x_2, \cdots, x_n \in X$. Then

$$\mu^*(q, x, p) = \vee\{\mu(q, x_1, q_1) \wedge \mu(q_1, x_2, q_2) \wedge \cdots \wedge \mu(q_{n-1}, x_n, p) | q_1, q_2, \cdots, q_{n-1} \in Q$$

Since $Im(\mu)$ is finite, $Im(\mu^*)$ is finite. Thus $(E(M), *)$ is a finite monoid.
□

A *fuzzy transformation semigroup (fts)* is triple (Q, S, ρ), where Q is a finite nonempty set, S is a finite semigroup and ρ is a fuzzy subset of $Q \times S \times Q$ such that

(i) $\rho(q, uv, p) = \vee\{\rho(q, u, v) \wedge \rho(r, v, p) | r \in Q\}$
$\forall u, v \in S$ and $\forall q, p \in Q$

(ii) If S contains the identity e,then $\rho(q, e, p) = 1$
if $q = p$ and $\rho(q, e, p) = 0$ if $q \neq p, \forall q, p \in Q$.

If, in addition, the following property holds, then (Q, S, ρ) is called *faithful.*

(iii) Let $u, v \in S$. If $\rho(q, u, p) = \rho(q, v, p) \forall q, p \in Q$, then $u = v$.

Let $M = (Q, X, \mu)$ be a ffsm. Then $(Q, E(M), \rho)$ is a fuzzy transformation semigroup which denoted by $FTS(M)$.

Examples

(1) Let $Q = \{0, 1\}$ and $X = \{x\}$. And define $\mu : Q \times X \times Q \to [0, 1]$ as follows.

Q	X	Q	μ
0	x	0	1
0	x	1	0
1	x	0	0.5
1	x	1	1

Then $(Q, E(M), \rho)$ is given as follows.

Q	$E(M)$	Q	ρ
0	$[x]$	0	1
0	$[x]$	1	0
1	$[x]$	0	0.5
1	$[x]$	1	1

(2) Let $Q = \{0, 1\}$ and $X = \{x\}$. And define $\mu : Q \times X \times Q \to [0, 1]$ as follows.

Q	X	Q	μ
0	x	0	0.3
0	x	1	0.6
1	x	0	1
1	x	1	0

Then $(Q, E(M), \rho)$ is given as follows.

Q	$E(M)$	Q	ρ
0	$[x]$	0	0.3
0	$[x]$	1	0.6
1	$[x]$	0	1
1	$[x]$	1	0
0	$[x^2]$	0	0.6
0	$[x^2]$	1	0.3
1	$[x^2]$	0	0.3
1	$[x^2]$	1	0.6
0	$[x^3]$	0	0.3
0	$[x^3]$	1	0.6
1	$[x^3]$	0	0.6
1	$[x^3]$	1	0.3

Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two fuzzy finite state machines. A pair (α, β) of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$, is called a *homomorphism*, written $(\alpha, \beta) : M_1 \to M_2$, if $\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p))$ for $\forall q, p \in Q_1$ and $\forall x \in X_1$.

The pair (α, β) is called a *strong homomorphism* if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \vee\{\mu_1(q, x, t) | t \in Q_1, \alpha(t) = \alpha(p)\}$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1$.

A homomorphism (strong homomorphism) $(\alpha, \beta) : M_1 \to M_2$ is called an *isomorphism (strong isomorphism)* if α and β are both one-one and onto.

Let (Q_1, S_1, ρ_1) and (Q_2, S_2, ρ_2) be two ftss. And let $f : Q_1 \to Q_2$ and $g : S_1 \to S_2$ be functions. A pair (f, g) of mappings is called a *homomorphism* from (Q_1, S_1, ρ_1) to (Q_2, S_2, ρ_2) if

(i) $g(xy) = g(x)g(y)$ for $\forall x, y \in S_1$,

(ii) If e_1 is the identity of S_1 and e_2 is the identity of S_2, then $g(e_1) = e_2$,

(iii) $\rho_1(q, x, p) \leq \rho_2(f(q), g(x), f(p))$ for $\forall q, p \in Q_1$, $\forall x \in S_1$.

The pair (f, g) is called a *strong homomorphism* if it satisfies (i),(ii) and

$$\rho_2(f(q), g(x), f(p)) = \vee\{\rho_1(q, x, t) | t \in Q_1, f(t) = f(p)\}$$

for $\forall q, p \in Q_1$ and $\forall x \in S_1$.

A homomorphism (strong homomorphism) $(f, g) : (Q_1, S_1, \rho_1) \to (Q_2, S_2, \rho_2)$ is called an *isomorphism (strong isomorphism)* if f and g are both one-one and onto.

Example

Let $M_1 = (Q_1, X_1, \mu_1)$ be a fuzzy finite state machine where $Q_1 = \{0, 1\}$ and $X_1 = \{x_1, y_1\}$. And define $\mu_1 : Q_1 \times X_1 \times Q_1 \to [0, 1]$ as follows.

Q_1	X_1	Q_1	μ_1
0	x_1	0	1
0	x_1	1	0.3
1	x_1	0	0.5
1	x_1	1	1
0	y_1	0	0
0	y_1	1	0.7
1	y_1	0	0.2
1	y_1	1	1

Let $M_2 = (Q_2, X_2, \mu_2)$ be a fuzzy finite state machine where $Q_2 = \{3, 4\}$ and $X_2 = \{x_2, y_2\}$. And define $\mu_2 : Q_2 \times X_2 \times Q_2 \to [0, 1]$ as follows.

Q_2	X_2	Q_2	μ_2
3	x_2	3	1
3	x_2	4	0.4
4	x_2	3	0.5
4	x_2	4	1
3	y_2	3	1
3	y_2	4	0.8
4	y_2	3	0.6
4	y_2	4	1

Define $\alpha : Q_1 \to Q_2$ by $\alpha(0) = 3$ and $\alpha(1) = 4$ and define $\beta : X_1 \to X_2$ by $\beta(x_1) = x_2$ and $\beta(y_1) = y_2$. Then $(\alpha, \beta) : M_1 \to M_2$ is clearly an isomorphism.

Remark 3.1.5.([10]) Let $M = (Q_1, X_1, \mu_1)$ and $M = (Q_2, X_2, \mu_2)$ be ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a strong homomorphism and let $\alpha : Q_1 \to Q_2$ be one-one. Then

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \mu_1(q, x, p)$$

for $\forall q, p \in Q$, $\forall x \in X_1$.

Proof. Let $t, p, q \in Q_1$ and $\forall x \in X_1$. Since α is one-one, if $\alpha(t) = \alpha(p)$, $t = p$. Therefore

$$\vee\{\mu_1(q, x, t) | t \in Q_1, \alpha(t) = \alpha(p)\} = \mu_1(q, x, t) = \mu_1(q, x, p).$$

□

Lemma 3.1.6.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a strong homomorphism. Then for $\forall q, r \in Q_1$ and $\forall x \in X_1$, if $\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$, then there exists $t \in Q_1$ such that $\mu_1(q, x, t) > 0$ and $\alpha(t) = \alpha(r)$.
Furthermore, for $\forall p \in Q_1$, if $\alpha(p) = \alpha(q)$, then $\mu_1(q, x, t) \geq \mu_1(p, x, r)$.

Proof. Let $\forall p, q, r \in Q_1$ and $\forall x \in X_1$.
And let $\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$. Then
$\mu_2(\alpha(q), \beta(x), \alpha(r)) = \vee\{\mu_1(q, x, s) | s \in Q_1, \alpha(s) = \alpha(r)\} > 0$.

Since Q_1 is finite, there exists $t \in Q_1$ such that $\alpha(t) = \alpha(r)$ and $\mu_1(q, x, t) = \vee\{\mu_1(q, x, s) | s \in Q_1, \alpha(s) = \alpha(r)\} > 0$.

Furthermore, suppose $\alpha(p) = \alpha(q)$. Then

$$\begin{aligned} \mu_1(q, x, t) &= \mu_2(\alpha(q), \beta(x), \alpha(r)) \\ &= \mu_2(\alpha(p), \beta(x), \alpha(r)) \\ &\geq \mu_1(p, x, r). \end{aligned}$$

□

Definition 3.1.7.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a homomorphism. Define $\beta^* : X_1^* \to X_2^*$ by

$$\beta^*(\lambda) = \lambda, \beta^*(ua) = \beta^*(u)\beta(a)$$

for $\forall u \in X_1^*$, $\forall a \in X_1$.

Lemma 3.1.8.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a homomorphism. Then

$$\beta^*(uv) = \beta^*(u)\beta^*(v)$$

for $\forall u, v \in X_1^*$.

Proof. Let $u, v \in X_1^*$ and $|v| = n$. If $n = 0$ then $v = \lambda$ and hence

$$\beta^*(uv) = \beta^*(u\lambda) = \beta^*(u) = \beta^*(u)\lambda = \beta^*(u)\beta^*(v).$$

Suppose now the result is true for $\forall y \in X_1^*$ such that $|y| = n - 1$, $n > 0$. Let $v = ya$ where $a \in X_1$ and $y \in X_1^*$ and $|y| = n - 1$, $n > 0$. Then

$$\begin{aligned} \beta^*(uv) &= \beta^*(uya) \\ &= \beta^*(uy)\beta(a) \\ &= \beta^*(u)\beta^*(y)\beta(a) \\ &= \beta^*(u)\beta^*(ya) \\ &= \beta^*(u)\beta^*(v). \end{aligned}$$

The result now follows by induction. □

Theorem 3.1.9.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a homomorphism. Then

$$\mu_1^*(q, x, p) \leq \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1^*$.

Proof. Let $p, q \in Q_1$ and $x \in X_1^*$. And let $|x| = n$.

Now Let $n = 0$. Then $x = \lambda$ and $\beta^*(x) = \beta^*(\lambda) = \lambda$. Now if $q = p$, then

$$\mu_1^*(q, \lambda, p) = 1 = \mu_2^*(\alpha(q), \lambda, \alpha(p)).$$

If $q \neq p$, then

$$\mu_1^*(q, \lambda, p) = 0 \leq \mu_2^*(\alpha(q), \lambda, \alpha(p)).$$

Now suppose the result is true for $y \in X_1^*$, $|y| = n-1$, $n > 0$. Let $x = ya$ where $y \in X_1^*$, $a \in X_1$ and $|y| = n - 1$. Then

$$\begin{aligned}
\mu_1^*(q, x, p) &= \mu_1^*(q, ya, p) \\
&= \vee\{\mu_1^*(q, y, r) \wedge \mu_1(r, a, p) | r \in Q_1\} \\
&\leq \vee\{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r)) \wedge \mu_2(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\} \\
&\leq \vee\{\mu_2^*(\alpha(q), \beta^*(y), r_2) \wedge \mu_2(r_2, \beta(a), \alpha(p)) | r_2 \in Q_2\} \\
&= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p)) \\
&= \mu_2^*(\alpha(q), \beta^*(ya), \alpha(p)) \\
&= \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)).
\end{aligned}$$

Therefore the result is true for $\forall n \geq 0$. □

Theorem 3.1.10.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a strong homomorphism. Then α is one-one if and only if

$$\mu_1^*(q, x, p) = \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1^*$.

Proof. Suppose α is one-one. Let $p, q \in Q_1$ and $x \in X_1^*$. And let $|x| = n$.

Now Let $n = 0$. Then $x = \lambda$ and $\beta^*(x) = \beta^*(\lambda) = \lambda$. Since α is one-one, $\alpha(q) = \alpha(p)$ if and only if $q = p$. Hence $\mu_1^*(q, \lambda, p) = 1$ if and only if $\mu_2^*(\alpha(q), \beta^*(\lambda), \alpha(p)) = 1$. Therefore the result is true for $n = 0$.

Now suppose the result is true for $y \in X_1^*$, $|y| = n-1$, $n > 0$. Let $x = ya$, $y \in X_1^*$, $a \in X_1$. Then

$$\begin{aligned}
\mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) &= \mu_2^*(\alpha(q), \beta^*(ya), \alpha(p)) \\
&= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p)) \\
&= \vee\{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r)) \wedge \mu_2(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\} \\
&= \vee\{\mu_1^*(q, y, r) \wedge \mu_1(r, a, p) | r \in Q_1\} \\
&= \mu_1^*(q, ya, p) \\
&= \mu_1^*(q, x, p).
\end{aligned}$$

Therefore the result is true for $\forall n \geq 0$.

Conversely, let $q, p \in Q_1$ and $\alpha(q) = \alpha(p)$. Then

$$1 = \mu_2^*(\alpha(q), \lambda, \alpha(p)) = \mu_1(q, \lambda, p).$$

Hence $q = p$, so α is one-one. □

Definition 3.1.11.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Let $\sim$ be an equivalence relation on Q. Then $\sim$ is called an *admissible relation* if and only if $\forall q, p, r \in Q$, $\forall a \in X$, if $p \sim q$ and $\mu(p, a, r) > 0$, then $\exists t \in Q$ such that $\mu(q, a, t) \geq \mu(p, a, r)$ and $t \sim r$.

Theorem 3.1.12.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Let $\sim$ be an equivalence relation on Q. Then $\sim$ is an admissible relation if and only if $\forall q, p, r \in Q$, $\forall x \in X^*$, if $p \sim q$ and $\mu^*(p, x, r) > 0$, then $\exists t \in Q$ such that $\mu^*(q, x, t) \geq \mu^*(p, x, r)$ and $t \sim r$.

Proof. Suppose $\sim$ is admissible. Let $\forall q, p \in Q$ and $p \sim q$. Let $x \in X^*$, $r \in Q$ and $\mu^*(p, x, r) > 0$. Suppose $|x| = n$. If $n = 0$, then $x = \lambda$. Thus if $\mu^*(p, x, r) > 0$, then $p = r$ and $\mu^*(p, x, p) = 1$. Now $\mu^*(q, x, q) = 1 = \mu^*(p, x, p)$ and $q = p$. Therefore the results is true for $n = 0$.

Now suppose the result is true for $\forall y \in X^*$, $|y| = n-1$, $n > 0$. Let $x = ya$, $y \in X^*$, $a \in X$ and $|y| = n-1$. Now

$$\mu^*(p, x, r) = \mu^*(p, ya, r) = \vee\{\mu^*(p, y, q_1) \wedge \mu^*(q_1, a, r) | q_1 \in Q\} > 0.$$

Let $s \in Q$ and $\mu^*(p, y, s) \wedge \mu^*(s, a, r) = \vee\{\mu^*(p, y, q_1) \wedge \mu^*(q_1, a, r) | q_1 \in Q\}$. Then $\mu^*(p, y, s) > 0$ and $\mu^*(s, a, r) > 0$. By the induction hypothesis, $\exists t_s \in Q$ such that $\mu^*(q, y, t_s) \geq \mu^*(p, y, s)$ and $t_s \sim s$. Now

$\mu(s,a,r) > 0$ and $t_s \sim s$. Since $\sim$ is admissible, $\exists t \in Q$ such that $\mu(t_s,a,t) \geq \mu(s,a,r)$ and $t \sim r$. Thus $\exists t \in Q$ such that $\mu^*(q,y,t_s) \wedge \mu(t_s,a,t) \geq \mu^*(p,y,s) \wedge \mu^*(s,a,r)$ and $t \sim r$. Hence

$$\begin{aligned} mu^*(p,x,r) &= \mu^*(p,y,s) \wedge \mu^*(s,a,r) \\ &\leq \mu^*(q,y,t_s) \wedge \mu(t_s,a,t) \\ &\leq \vee\{\mu^*(q,y,r_1) \wedge \mu(r_1,a,t) | r_1 \in Q\} \\ &= \mu^*(q,ya,t) \\ &= \mu^*(q,x,t) \end{aligned}$$

and $t \sim r$. the results now follows by induction.

The converse is trivial. □

Definition 3.1.13.([10]) Let $M = (Q,X,\mu)$ be a ffsm. Let $\sim$ be an admissible relation on Q. Let $[q]$ denote the equivalence class of $q \in Q$. Let $\tilde{Q} = Q/\sim = \{[q] | q \in Q\}$. Define the fuzzy subset $\tilde{\mu}$ of $\tilde{Q} \times X \times \tilde{Q}$ by

$$\tilde{\mu}([q],x,[p]) = \vee\{\mu(q,x,t) | t \in [p]\}$$

for $\forall q,p \in Q$, $x \in X$.

Remark 3.1.14.([10]) Let $M = (Q,X,\mu)$ be a ffsm. Then $(\tilde{Q},X,\tilde{\mu})$ is a ffsm.

Proof. Let $q,q',p,p' \in Q$ such that $[q] = [q']$ and $[p] = [p']$. Let $x,y \in X$ such that $x = y$. Now

$$\tilde{\mu}([q],x,[p]) = \vee\{\mu(q,x,r) | r \in [p]\}$$

and

$$\tilde{\mu}([q'],y,[p']) = \tilde{\mu}([q'],x,[p']) = \vee\{\mu(q',x,t) | t \in [p']\}.$$

Let $r \in [p]$ be such that $\mu(q,x,r) > 0$. Then since $\sim$ is admissible, $\exists t \in Q$ such that $\mu(q',x,t) \geq \mu(q,x,r) > 0$ and $t \sim r$. Now since $t \sim r$, $t \in [p] = [p']$. Thus $\exists t \in [p']$ such that $\mu(q',x,t) \geq \mu(q,x,r) > 0$. Similarly, if there exists $t \in [p']$ such that $\mu(q',x,t) > 0$, then $\exists r \in [p]$ such that $\mu(q,x,r) \geq \mu(q',x,t) > 0$. Hence

$$\tilde{\mu}([q],x,[p]) = \tilde{\mu}([q'],x,[p']).$$

Now let $r \in [p]$ be such that $\mu(q,x,r) = 0$. Then $\tilde{\mu}([q],x,[p]) = 0$. Hence

$$\tilde{\mu}([q'],x,[p']) \geq \tilde{\mu}([q],x,[p]).$$

Similarly, let $t \in [p']$ be such that $\mu(q', x, t) = 0$. Then

$$\tilde{\mu}([q], x, [p]) \geq \tilde{\mu}([q'], x, [p']) = 0.$$

Therefore

$$\tilde{\mu}([q'], x, [p']) = \tilde{\mu}([q], x, [p]).$$

Thus $\tilde{\mu}$ is well-defined. Hence $(\tilde{Q}, X, \tilde{\mu})$ is a ffsm. □

Definition 3.1.15.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Define $\underline{\alpha} : Q \to \tilde{Q}$ by $\underline{\alpha}(q) = [q]$ for $\forall q \in Q$.

Remark 3.1.16.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Let $\beta : X \to X$ be the identity map. Then $(\underline{\alpha}, \beta) : (Q, X, \mu) \to (\tilde{Q}, X, \tilde{\mu})$ is a homomorphism.

Proof. Clearly, $\underline{\alpha} : Q \to \tilde{Q}$ is an onto mapping. Let $q, t \in Q$ and $x \in X$. Then

$$\begin{aligned}
\tilde{\mu}(\underline{\alpha}(q), x, \underline{\alpha}(t)) &= \tilde{\mu}([q], x, [t]) \\
&= \vee\{\mu(q, x, r) | r \in [t]\} \\
&\geq \mu(q, x, t).
\end{aligned}$$

Hence $(\underline{\alpha}, \beta) : (Q, X, \mu) \to (\tilde{Q}, X, \tilde{\mu})$ is a homomorphism. □

Definition 3.1.17.([10]) Let $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two ffsms. Let $\alpha : M_1 \to M_2$ be a strong homomorphism.

The *kernel* of α, denoted Ker α, is defined to be the set

$$Ker\, \alpha = \{(p, q) | \alpha(p) = \alpha(q)\}.$$

Lemma 3.1.18.([10]) Let $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two ffsms. Let $\alpha : M_1 \to M_2$ be a strong homomorphism.

Then Ker α is an admissible relation.

Proof. Clearly, Ker α is an equivalence relation. Let $p, q \in Q_1$ and $(p, q) \in$ Ker α. Then $\alpha(p) = \alpha(q)$. Let $a \in X$, $r \in Q_1$ and $\mu_1(p, a, r) > 0$. Since α is a strong homomorphism, then

$$\mu_2(\alpha(q), a, \alpha(r)) = \mu_2(\alpha(p), a, \alpha(r)) \geq \mu_1(p, a, r) > 0.$$

By Lemma 1.6., there exists $t \in Q_1$ such that $\mu_1(q, a, t) \geq \mu_1(p, a, r) > 0$ and $\alpha(t) = \alpha(r)$. Since $\alpha(t) = \alpha(r)$, $(t, r) \in$ Ker α. Thus Ker α is admissible. $\Box$

Theorem 3.1.19.([10]) Let $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two ffsms. Let $\alpha : M_1 \to M_2$ be an onto strong homomorphism.

Then there exists an isomorphism $\gamma : (Q_1/(Ker\alpha), X, \widetilde{\mu_1}) \to (Q_2, X, \mu_2)$ such that $\alpha = \gamma \circ \alpha$.

Proof. Define $\gamma : Q_1/(Ker\,\alpha) \to Q_2$ by $\gamma([q]) = \alpha(q)$ for $q \in Q_1$.

Let $p, q \in Q_1$ be such that $[p] = [q]$. Since $(p, q) \in Ker\,\alpha$, $\alpha(p) = \alpha(q)$. Hence $\gamma([p]) = \gamma([q])$, so γ is well-defined.

Let $p, q \in Q_1$ and $x \in X$. Then

$$\begin{aligned}
\widetilde{\mu_1}([q], x, [p]) &= \vee\{\mu_1(q, x, r) | r \in [p]\} \\
&= \vee\{\mu_1(q, x, r) | \alpha(r) = \alpha(p), r \in Q_1\} \\
&= \mu_2(\alpha(q), x, \alpha(p)) \\
&= \mu_2(\gamma([q]), x, \gamma([p])).
\end{aligned}$$

Therefore γ is a homomorphism.

Let $[p], [q] \in Q_1/(Ker\,\alpha)$ such that $\gamma([p]) = \gamma([q])$. Then $\alpha(q) = \alpha(p)$, i.e., $(q, p) \in Ker\,\alpha$. Hence $[p] = [q]$, so γ is one-one.

Let $q_2 \in Q_2$. As α is onto, there exists $q \in Q_1$ such that $q_2 = \alpha(q)$. Since $\alpha(q) = \gamma([q])$, there exists $[q] \in Q_1/(Ker\,\alpha)$ such that $q_2 \in \gamma([q])$. Hence γ is onto. Thus γ is an isomorphism. $\Box$

Theorem 3.1.20.([10]) Let $M = (Q, X, \mu)$ be a ffsm. Then $(Q, E(M), \rho)$ is a faithful fts where $\rho(q, [x], p) = \mu^*(q, x, p)$ for $\forall q, p \in Q$, $x \in X^*$.

Proof. By Theorem 3.1.4., $E(M)$ is a finite monoid. Clearly, ρ is well-defined. Let $q, p \in Q$ and $[x], [y] \in E(M)$. Then

$$\begin{aligned}
\rho(q, [x] * [y], p) &= \rho(q, [xy], p) \\
&= \mu^*(q, xy, p) \\
&= \vee\{\mu^*(q, x, r) \wedge \mu^*(r, y, p) | r \in Q\} \\
&= \vee\{\rho(q, [x], r) \wedge \rho(r, [y], p) | r \in Q\}.
\end{aligned}$$

Therefore The condition (i) to be a fts holds.

Let $q, p \in Q$. If $p = q$, Then $\mu^*(q, \lambda, p) = 1$. Hence $\rho(q, [\lambda], p) = \mu^*(q, \lambda, p) = 1$. If $p \neq q$, Then $\mu^*(q, \lambda, p) = 0$. Hence $\rho(q, [\lambda], p) = \mu^*(q, \lambda, p) = 0$. Therefore The condition (ii) to be a fts holds.

Let $q, p \in Q$ such that $\rho(q, [x], p) = \rho(q, [y], p)$. Then $\mu^*(q, x, p) = \mu^*(q, y, p)$. Thus $x \equiv y$ and $[x] = [y]$. Hence $(Q, E(M), \rho)$ is a faithful fts. □

Let $M = (Q, X, \mu)$ be a ffsm. Then by Theorem 3.1.20., $(Q, E(M), \rho)$ is a faithful fts which we denote by $FTS(M)$.

Remark 3.1.21.([10]) Let S be a monoid. Let $A = (Q, S, \delta)$ be a faithful fts. Define the ffsm $M = (Q, X, \mu)$ by taking $\mu = \delta$. Consider $FTS(M) = (Q, E(M), \rho)$, where $E(M) = S^*/\sim$ and $\rho(q, [u], p) = \mu^*(q, u, p)$, $u \in S$. Let e be the identity element of S and λ the empty word in S^*.

Then $[e] = [\lambda]$.

Proof. Let $q, p \in Q$. Then $\rho(q, [e], p) = \mu^*(q, e, p) = \delta(q, e, p)$. If $p = q$, then $\rho(q, [e], p) = 0$. If $p \neq q$, then $\rho(q, [e], p) = 1$. Therefore $\rho(q, [e], p) = \rho(q, [\lambda], p)$ for $\forall q, p \in Q$. □

Theorem 3.1.22.([10]) Let S be a monoid. Then

$$FTS(M) \cong A = (Q, S, \delta).$$

Proof. Define $f : Q \to Q$ by $f(q) = q$ for $\forall q \in Q$. Then clearly $f : Q \to Q$ is one-one and onto. Define $g : S \to E(M)$ by $g(x) = [x]$ for $\forall x \in S$.

Let $x, y \in S$ such that $g(x) = g(y)$. Then $[x] = [y]$. Thus $\mu^*(q, x, p) = \mu^*(q, y, p)$ for $\forall q, p \in Q$. Hence $\mu(q, x, p) = \mu(q, y, p)$ for $\forall q, p \in Q$. This implies that $\delta(q, x, p) = \delta(q, y, p)$ for $\forall q, p \in Q$. Since A is faithful, $x = y$. Hence g is injective.

Let $\cdot$ denote the binary operation of the semigroup S. Let $a, b \in S$. Then $a \cdot b \in S$ and $ab \in S^*$. Let $q, p \in Q$. Then

$$\begin{aligned} \mu^*(q, a \cdot b, p) &= \delta(q, a \cdot b, p) \\ &= \vee\{\delta(q, a, r) \wedge \delta(r, b, p) | r \in Q\} \\ &= \vee\{\mu(q, a, r) \wedge \mu(r, b, p) | r \in Q\} \end{aligned}$$

$$= \quad \mu^*(q, ab, p).$$

Hence $[a \cdot b] = [ab]$. Thus

$$g(ab) = g(a \cdot b) = [a \cdot b] = [ab] = [a][b] = g(a)g(b).$$

By induction it can be shown that if $c_i inS$, $1 \leq i \leq n$, then $[c_1 \cdot c_2 \cdot \ldots \cdot c_n] = [c_1 c_2 \ldots c_n]$.

If $i = 1$, the result is true clearly. Suppose the result is true for $c_i inS$, $1 \leq i \leq n-1$. Let $b_i inS$, $1 \leq i \leq n$. Now

$$\begin{aligned} [b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1} \cdot b_n] &= [(b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) \cdot b_n] \\ &= [b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}][b_n] \\ &= [b_1 b_2 \ldots b_{n-1}][b_n] \\ &= [(b_1 b_2 \ldots b_{n-1}) b_n] \\ &= [(b_1 b_2 \ldots b_{n-1} b_n]. \end{aligned}$$

Thus if $c_i inS$, $1 \leq i \leq n$, then $[c_1 \cdot c_2 \cdot \ldots \cdot c_n] = [c_1 c_2 \ldots c_n]$.

Let $[u] \in E(M)$. If $u = \lambda$, then $[\lambda] = [e]$ and $g(e) = [\lambda]$.

Suppose $u = a_1 a_2 \ldots a_n$, $a_i \in S$, $1 \leq i \leq n$. Then

$$g(a_1 \cdot a_2 \cdot \ldots \cdot a_n) = [a_1 \cdot a_2 \cdot \ldots \cdot a_n] = [a_1 a_2 \ldots a_n] = [u].$$

Thus g is surjective. Finally,

$$\rho(f(q), g(x), f(p)) = \rho(q, [x], p) = \mu^*(q, x, p) = \mu(q, x, p) = \delta(q, x, p).$$

for $\forall q, p \in Q$, $\forall x \in S$. Thus $FTS(M) \cong A = (Q, S, \delta)$. □

Theorem 3.1.23.([10]) Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a strong homomorphism with α one-one and onto.

Then there exists a strong homomorphism $(f_\alpha, g_\beta) : FTS(M_1) \to FTS(M_2)$.

Proof. Let $FTS(M_1) = (Q_1, E(M_1), \rho_1)$ and $FTS(M_2) = (Q_2, E(M_2), \rho_2)$ Define $f_\alpha : Q_1 \to Q_2$ by $f_\alpha(q) = \alpha(q)$ for $\forall q \in Q_1$. Define $g_\beta : E(M_1) \to E(M_2)$ by $g_\beta([x]) = [\beta^*(x)]$ for $\forall [x] \in E(M_1)$.

Let $[x], [y] \in E(M_1)$ such that $[x] = [y]$. Then $\mu_1^*(q, x, p) = \mu_1^*(q, y, p)$ for $q, p \in Q_1$. Since α is one-one and onto, and (α, β) is a strong homomorphism, by Theorem 3.1.10., then

$$\mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \quad = \quad \mu_1^*(q, x, p)$$

$$\begin{aligned} &= \mu_1^*(q, y, p) \\ &= \mu_2^*(\alpha(q), \beta^*(y), \alpha(p)) \end{aligned}$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1^*$. As α is onto,

$$\mu_2^*(q_2, \beta^*(x), p_2) = \mu_2^*(q_2, \beta^*(y), p_2)$$

for $\forall q_2, p_2 \in Q_2$. Hence $[\beta^*(x)] = [\beta^*(y)]$.

Let $[x], [y] \in E(M_1)$. By Lemma 3.1.8.,

$$\begin{aligned} g_\beta([x] * [y]) &= g_\beta([xy]) \\ &= [\beta^*(xy)] \\ &= [\beta^*(x)\beta^*(y)] \\ &= [\beta^*(x)] * [\beta^*(y)] \\ &= g_\beta([x]) * g_\beta([y]). \end{aligned}$$

Let $[\lambda] \in E(M_1)$. Then

$$g_\beta([\lambda]) = [\beta^*(\lambda)] = [\lambda].$$

Let $[x] \in E(M_1)$ and $q, p \in Q_1$. By Theorem 1.10. and 1.20, then

$$\begin{aligned} \rho_1(q, [x], p) &= \mu_1^*(q, x, p) \\ &= \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \\ &= \rho_2(f_\alpha(q), g_\beta([x]), f_\alpha(p)). \end{aligned}$$

As α is one-one,

$$\begin{aligned} \rho_2(f_\alpha(q), g_\beta([x]), f_\alpha(p)) &= \vee\{\rho_1(q, [x], t) | t, \in Q_1, f_\alpha(t) = f_\alpha(p)\} \\ &= \rho_1(q, [x], p). \end{aligned}$$

Hence (f_α, g_β) is a strong homomorphism. □

3.2. Semigroups of Fuzzy Finite Switchboard State Machine

Now we introduce a concept of fuzzy finite switchboard state machine.

Definition 3.2.1. Let $M = (Q, X, \mu)$ be a fuzzy finite state machine. M is called *switching* if and only if

$$\mu(q, a, p) = \mu(p, a, q)$$

for $\forall q, p \in Q$, $\forall a \in X$.

M is called *commutative* if and only if

$$\mu(q, ab, p) = \mu(q, ba, p)$$

for $\forall q, p \in Q$, $\forall a, b \in X$.

If M is switching and commutative, then M is called a *fuzzy finite switchboard state machine (ffssm)*.

Lemma 3.2.2. Let $M = (Q, X, \mu)$ be a commutative ffsm. Then

$$\mu^*(q, xa, p) = \mu^*(q, ax, p)$$

for $\forall q, p \in Q$, $\forall a \in X$, $\forall x \in X^*$.

Proof. Let $a \in X$ and $x \in X^*$, and $|x| = n$. If $n = 0$, then $x = \lambda$. Hence

$$\begin{aligned}\mu^*(q, xa, p) = \mu^*(q, \lambda a, p) &= \mu^*(q, a, p) = \mu^*(q, a\lambda, p) \\ &= \mu^*(q, ax, p).\end{aligned}$$

Suppose now the result is true for $\forall u \in X^*$ such that $|u| = n-1$, $n > 0$. Let $x = ub$ where $b \in X$ and $x \in X^*$. Then

$$\begin{aligned}\mu^*(q, xa, p) = \mu^*(q, uba, p) &= \vee\{\mu^*(q, u, r) \wedge \mu^*(r, ba, p) | r \in Q\} \\ &= \vee\{\mu^*(q, u, r) \wedge \mu^*(r, ab, p) | r \in Q\} \\ &= \mu^*(q, uab, p) \\ &= \vee\{\mu^*(q, ua, r) \wedge \mu(r, b, p) | r \in Q\} \\ &= \vee\{\mu^*(q, au, r) \wedge \mu(r, b, p) | r \in Q\} \\ &= \mu^*(q, aub, p) \\ &= \mu^*(q, ax, p).\end{aligned}$$

The result now follows by induction. □

Lemma 3.2.3. Let $M = (Q, X, \mu)$ be a ffssm. Then

$$\mu^*(q, x, p) = \mu^*(p, x, q)$$

for $\forall q, p \in Q$, $\forall x \in X^*$.

Proof. Let $x \in X^*$, and $|x| = n$. If $n = 0$, then $x = \lambda$. Hence

$$\mu^*(q, x, p) = \mu^*(q, \lambda, p) = \mu^*(p, \lambda, q) = \mu^*(p, x, q).$$

Suppose now the result is true for $\forall u \in X^*$ such that $|u| = n-1$, $n > 0$. Let $x = ua$ where $a \in X$ and $x \in X^*$. Then

$$\begin{aligned}
\mu^*(q,x,p) = \mu^*(q,ua,p) &= \vee\{\mu^*(q,u,r) \wedge \mu(r,a,p) | r \in Q\} \\
&= \vee\{\mu^*(r,u,q) \wedge \mu(p,a,r) | r \in Q\} \\
&= \vee\{\mu^*(r,u,q) \wedge \mu^*(p,a,r) | r \in Q\} \\
&= \vee\{\mu^*(p,a,r) \wedge \mu^*(r,u,q) | r \in Q\} \\
&= \mu^*(p,au,q) \\
&= \mu^*(p,ua,q) \qquad \text{(by Lemma 3.2.2)} \\
&= \mu^*(p,x,q).
\end{aligned}$$

The result now follows by induction. □

Lemma 3.2.4. Let $M = (Q, X, \mu)$ be a ffssm. Then

$$\mu^*(q,xy,p) = \mu^*(q,yx,p)$$

for $\forall q, p \in Q$, $\forall x, y \in X^*$.

Proof. Let $y \in X^*$, and $|y| = n$. If $n = 0$, then $y = \lambda$. Hence

$$\begin{aligned}
\mu^*(q,xy,p) = \mu^*(q,x\lambda,p) &= \mu^*(q,x,p) = \mu^*(q,\lambda x,p) \\
&= \mu^*(q,yx,p).
\end{aligned}$$

Suppose now the result is true for $\forall u \in X^*$ such that $|u| = n-1$, $n > 0$. Let $y = ua$ where $a \in X$ and $u \in X^*$. Then

$$\begin{aligned}
\mu^*(q,xy,p) &= \mu^*(q,xua,p) \\
&= \vee\{\mu^*(q,xu,r) \wedge \mu(r,a,p) | r \in Q\} \\
&= \vee\{\mu^*(q,ux,r) \wedge \mu^*(r,a,p) | r \in Q\} \\
&= \vee\{\mu^*(r,ux,q) \wedge \mu(p,a,r) | r \in Q\} \quad \text{(Lemma 3.2.3)} \\
&= \vee\{\mu(p,a,r) \wedge \mu^*(r,ux,q) | r \in Q\} \\
&= \mu^*(p,aux,q) \\
&= \vee\{\mu^*(p,au,r) \wedge \mu^*(r,x,q) | r \in Q\} \\
&= \vee\{\mu^*(p,ua,r) \wedge \mu^*(r,x,q) | r \in Q\} \quad \text{(Lemma 3.2.2)} \\
&= \mu^*(p,uax,q) \\
&= \mu^*(q,uax,p) \quad \text{(Lemma 3.2.3)} \\
&= \mu^*(q,yx,p).
\end{aligned}$$

The result now follows by induction. □

Definition 3.2.5. Let $A = (Q, S, \rho)$ be a fuzzy transformation semigroup. A is called *switching* if and only if

$$\rho(q, a, p) = \rho(p, a, q)$$

for $\forall q, p \in Q$, $\forall a \in S$.

A is called *commutative* if and only if

$$\rho(q, ab, p) = \rho(q, ba, p)$$

for $\forall q, p \in Q$, $\forall a, b \in S$.

If A is switching and commutative, then A is called a *fuzzy switchboard transformation semigroup (fsts).*

Proposition 3.2.6. Let $M = (Q, X, \mu)$ be a ffssm. Then $(Q, E(M), \rho)$ is a faithful fsts, where $\rho(q, [x], p) = \mu^*(q, x, p)$, for $\forall q, p \in Q$, $\forall x \in X^*$.

Proof. By Theorem 3.1.20, clearly $(Q, E(M), \rho)$ is a faithful fts.

Let $[x] \in E(M)$ for $x \in X^*$, and $p, q \in Q$. Then

$$\begin{aligned} \rho(q, [x], p) &= \mu^*(q, x, p) \\ &= \mu^*(p, x, q) \qquad \text{(Lemma 3.2.3)} \\ &= \rho(p, [x], q). \end{aligned}$$

Hence $(Q, E(M), \rho)$ is switching.

Let $[x], [y] \in E(M)$ for $x, y \in X^*$, and $p, q \in Q$. Then

$$\begin{aligned} \rho(p, [x] * [y], q) &= \rho(p, [xy], q) \\ &= \mu^*(p, xy, q) \\ &= \mu^*(p, yx, q) \qquad \text{(Lemma 3.2.4)} \\ &= \rho(p, [yx], q) \\ &= \rho(p, [y] * [x], q). \end{aligned}$$

Hence $(Q, E(M), \rho)$ is commutative. □

3.3. Homomorphisms

Proposition 3.3.1. Let $M_1 = (Q_1, X_1, \mu_1)$ be a commutative ffsm, and $M_2 = (Q_2, X_2, \mu_2)$ a ffsm. Let $(\alpha, \beta) : M_1 \to M_2$ be an onto strong homomorphism. Then M_2 is a commutative ffsm.

Proof. Let $\forall q_2, p_2 \in Q_2$. As $\alpha : Q_1 \to Q_2$ is onto mapping, there exists $q_1, p_1 \in Q_1$ such that $\alpha(q_1) = q_2$ and $\alpha(p_1) = p_2$.

Let $\forall a_2, b_2 \in X_2$. As $\beta : X_1 \to X_2$ is onto mapping, there exists $a_1, b_1 \in X_1$ such that $\beta(a_1) = a_2$ and $\beta(b_1) = b_2$.

As M_1 is commutative, then

$$\begin{aligned}
\mu_2^*(q_2, a_2b_2, p_2) &= \mu_2^*(\alpha(q_1), \beta(a_1)\beta(b_1), \alpha(p_1)) \\
&= \mu_2^*(\alpha(q_1), \beta(a_1b_1), \alpha(p_1)) \\
&= \vee\{\mu_1^*(q_1, a_1b_1, t_1) | t_1 \in Q_1, \alpha(t_1) = \alpha(p_1)\} \\
&= \vee\{\mu_1^*(q_1, b_1a_1, t_1) | t_1 \in Q_1, \alpha(t_1) = \alpha(p_1)\} \\
&= \mu_2^*(\alpha(q_1), \beta(b_1a_1), \alpha(p_1)) \\
&= \mu_2^*(\alpha(q_1), \beta(b_1)\beta(a_1), \alpha(p_1)) \\
&= \mu_2^*(q_2, b_2a_2, p_2).
\end{aligned}$$

Thus M_2 is commutative. □

Definition 3.3.2. Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ be mappings. A pair (α, β) is called a *switching homomorphism* if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \vee\{\mu_1(s, x, t) | s, t \in Q_1, \alpha(t) = \alpha(p), \alpha(s) = \alpha(q)\}$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1$.

Proposition 3.3.3. Let $M_1 = (Q_1, X_1, \mu_1)$ be a ffssm, and $M_2 = (Q_2, X_2, \mu_2)$ be a ffsm. Let $(\alpha, \beta) : M_1 \to M_2$ be an onto switching homomorphism. Then M_2 is a ffssm.

Proof. Let $\forall q_2, p_2 \in Q_2$. As $\alpha : Q_1 \to Q_2$ is onto mapping, there exists $q_1, p_1 \in Q_1$ such that $\alpha(q_1) = q_2$ and $\alpha(p_1) = p_2$.

Let $\forall a_2, b_2 \in X_2$. As $\beta : X_1 \to X_2$ is onto mapping, there exists $a_1, b_1 \in X_1$ such that $\beta(a_1) = a_2$ and $\beta(b_1) = b_2$.

As M_1 is commutative, then

$$\begin{aligned}
\mu_2^*(q_2, a_2b_2, p_2) &= \mu_2^*(\alpha(q_1), \beta(a_1)\beta(b_1), \alpha(p_1)) \\
&= \mu_2^*(\alpha(q_1), \beta(a_1b_1), \alpha(p_1)) \\
&= \vee\{\mu_1^*(s_1, a_1b_1, t_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\
&= \vee\{\mu_1^*(s_1, b_1a_1, t_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\
&= \mu_2^*(\alpha(q_1), \beta(b_1a_1), \alpha(p_1))
\end{aligned}$$

$$\begin{aligned} &= \mu_2^*(\alpha(q_1), \beta(b_1)\beta(a_1), \alpha(p_1)) \\ &= \mu_2^*(q_2, b_2a_2, p_2). \end{aligned}$$

Hence M_2 is commutative.

As M_1 is switching, then

$$\begin{aligned} \mu_2(q_2, a_2, p_2) &= \mu_2(\alpha(q_1), \beta(a_1), \alpha(p_1)) \\ &= \vee\{\mu_1(s_1, a_1, t_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\ &= \vee\{\mu_1(t_1, a_1, s_1) | s_1, t_1 \in Q_1, \alpha(s_1) = \alpha(q_1), \alpha(t_1) = \alpha(p_1)\} \\ &= \mu_2(\alpha(p_1), \beta(a_1), \alpha(q_1)) \\ &= \mu_2(p_2, a_2, q_2). \end{aligned}$$

Hence M_2 is switching.

Thus M_2 is a ffssm. □

Proposition 3.3.4. Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a switching homomorphism. Then for $\forall q, r \in Q_1$ and $\forall x \in X_1$, if $\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$, then there exists $t, v \in Q_1$ such that $\mu_1(v, x, t) > 0$ and $\alpha(t) = \alpha(r)$,$\alpha(v) = \alpha(q)$. Furthermore, for $\forall p, k \in Q_1$, if $\alpha(p) = \alpha(q)$,$\alpha(k) = \alpha(r)$, then $\mu_1(v, x, t) \geq \mu_1(p, x, k)$.

Proof. Let $\forall p, q, r \in Q_1$ and $\forall x \in X_1$. And let
$\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$. Then
$\mu_2(\alpha(q), \beta(x), \alpha(r))$
$= \vee\{\mu_1(s, x, w) | s, w \in Q_1, \alpha(w) = \alpha(r), \alpha(s) = \alpha(q)\} > 0$.
Since Q_1 is finite, there exists $t, v \in Q_1$ such that
$\alpha(t) = \alpha(r)$, $\alpha(v) = \alpha(q)$ and
$\mu_1(v, x, t)$
$= \vee\{\mu_1(s, x, w) | s, w \in Q_1, \alpha(w) = \alpha(r), \alpha(s) = \alpha(q)\} > 0$.

Furthermore, suppose $\alpha(p) = \alpha(q)$ and $\alpha(k) = \alpha(r)$. Then

$$\begin{aligned} \mu_1(v, x, t) &= \mu_2(\alpha(q), \beta(x), \alpha(r)) \\ &= \mu_2(\alpha(p), \beta(x), \alpha(k)) \\ &\geq \mu_1(p, x, k). \quad \square \end{aligned}$$

Remark 3.3.5. Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be a switching homomorphism with α one-one. Then for $\forall q, p \in Q_1$ and $\forall x \in X_1$,

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \mu_1(q, x, p).$$

Proposition 3.3.6. Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two ffsms. Let $(\alpha, \beta) : M_1 \to M_2$ be an onto switching homomorphism. Then α is one-one if and only if

$$\mu_1^*(q, x, p) = \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1^*$.

Proof. Let $p, q \in Q_1$ and $x \in X_1^*$. And let $|x| = n$.

Now Let $n = 0$. Then $x = \lambda$ and $\beta^*(x) = \beta^*(\lambda) = \lambda$. Since α is one-one, $\alpha(q) = \alpha(p)$ if and only if $q = p$. Hence $\mu_1^*(q, \lambda, p) = 1$ if and only if $\mu_2^*(\alpha(q), \beta^*(\lambda), \alpha(p)) = 1$. Therefore the result is true for $n = 0$.

Now suppose the result is true for $y \in X_1^*$, $|y| = n - 1$, $n > 0$. Let $x = ya$, $a \in X_1$. As α is onto, and by the inductive assumption and Remark 3.3.5, then

$$\begin{aligned}
\mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) &= \mu_2^*(\alpha(q), \beta^*(ya), \alpha(p)) \\
&= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p)) \\
&= \vee\{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r)) \wedge \mu_2(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\} \\
&= \vee\{\mu_1^*(q, y, r) \wedge \mu_1(r, a, p) | r \in Q_1\} \\
&= \mu_1^*(q, ya, p) \\
&= \mu_1^*(q, x, p).
\end{aligned}$$

Therefore the result is true for $\forall n \geq 0$.

Conversely, let $q, p \in Q_1$ and $\alpha(q) = \alpha(p)$. Then

$$1 = \mu_2^*(\alpha(q), \lambda, \alpha(p)) = \mu_1(q, x, p).$$

Hence $q = p$, so α is one-one. □

Definition 3.3.7. Let $M_1 = (Q_1, X, \mu_1)$ and $M_2 = (Q_2, X, \mu_2)$ be two ffsms. Let $\alpha : M_1 \to M_2$ be a switching homomorphism.

The *switching kernel* of α, denoted SKer α, is defined to be the set

$$SKer\,\alpha = \{(p,q)|\alpha(p) = \alpha(q)\}.$$

Definition 3.3.8. Let $M = (Q, X, \mu)$ be a ffsm, and let $\sim$ be an equivalence relation on Q. For $q \in Q$, let $[q]$ denote the equivalence class of q. Let $\tilde{Q} = Q/\sim = \{[q]|q \in Q\}$.
Define the fuzzy subset $\bar{\mu}$ of $\tilde{Q} \times X \times \tilde{Q}$ by

$$\bar{\mu}([q], x, [p]) = \vee\{\mu(s, x, t)|s \in [q], t \in [p]\}$$

for $\forall q, p, \in Q,\ x \in X$.

Since clearly $\bar{\mu}$ is single-valued, $(\tilde{Q}, X, \bar{\mu})$ is a ffsm.

Proposition 3.3.9. Let $M = (Q, X, \mu)$ be a ffssm, and let $\sim$ be an equivalence relation on Q. Then $(\tilde{Q}, X, \bar{\mu})$ is a ffssm.

Proof. Let $[p], [q] \in \tilde{Q}$ and $a \in X$. Since M is switching, then

$$\begin{aligned}
\bar{\mu}([p], a, [q]) &= \vee\{\mu(s, x, t)|s \in [p], t \in [q]\} \\
&= \vee\{\mu(t, x, s)|s \in [p], t \in [q]\} \\
&= \bar{\mu}([q], a, [p]).
\end{aligned}$$

Hence $(\tilde{Q}, X, \bar{\mu})$ is switching.

Let $a, b \in X$. Since M is commutative, then

$$\begin{aligned}
\bar{\mu}^*([p], ab, [q]) &= \vee\{\bar{\mu}([p], a, [r]) \wedge \bar{\mu}([r], b, [q])|[r] \in \tilde{Q}\} \\
&= \vee\{(\vee\{\mu(t, a, s)|t \in [p], s \in [r]\}) \wedge (\vee\{\mu(s, b, k)|s \in [r], k \in [q]\})|[r] \\
&= \vee\{\vee\{\mu(t, a, s) \wedge \mu(s, b, k)|s \in [r], [r] \in \tilde{Q}\}|t \in [p], k \in [q]\} \\
&= \vee\{\mu^*(t, ab, k)|t \in [p], k \in [q]\} \\
&= \vee\{\mu^*(t, ba, k)|t \in [p], k \in [q]\} \\
&= \vee\{\vee\{\mu(t, b, s) \wedge \mu(s, a, k)|s \in [r], [r] \in \tilde{Q}\}|t \in [p], k \in [q]\} \\
&= \vee\{(\vee\{\mu(t, b, s)|t \in [p], s \in [r]\}) \wedge (\vee\{\mu(s, a, k)|s \in [r], k \in [q]\})|[r] \\
&= \vee\{\bar{\mu}([p], b, [r]) \wedge \bar{\mu}([r], a, [q])|[r] \in \tilde{Q}\} \\
&= \bar{\mu}^*([p], ba, [q]).
\end{aligned}$$

Hence $(\tilde{Q}, X, \bar{\mu})$ is commutative, so $(\tilde{Q}, X, \bar{\mu})$ is a ffssm. $\square$

Proposition 3.3.10. Let $M_1 = (Q_1, X, \mu_1)$ be a ffssm, and $M_2 = (Q_2, X, \mu_2)$ a ffsm. Let $\alpha : M_1 \to M_2$ be an onto switching homomorphism. Then there exists a switching isomorphism $\gamma : (Q_1/(SKer\,\alpha), X, \bar{\mu}_1) \to (Q_2, X, \mu_2)$, and both $(Q_1/(SKer\,\alpha), X, \bar{\mu}_1)$ and (Q_2, X, μ_2) are ffssms.

Proof. Define $\gamma : Q_1/(SKer\,\alpha) \to Q_2$ by $\gamma([q]) = \alpha(q)$ for $q \in Q_1$.

Let $p, q \in Q_1$ such that $[p] = [q]$. Since $(p, q) \in SKer\,\alpha$, $\alpha(p) = \alpha(q)$. Hence $\gamma([p]) = \gamma([q])$, so γ is well-defined.

Let $[p], [q] \in Q_1/(SKer\,\alpha)$ and $x \in X$. Then

$$\begin{aligned}
\bar{\mu}_1([q], x, [p]) &= \vee\{\mu_1(s, x, r) | s \in [q], r \in [p]\} \\
&= \vee\{\mu_1(s, x, r) | \alpha(s) = \alpha(q), \alpha(r) = \alpha(p), r, s \in Q_1\} \\
&= \mu_2(\alpha(q), x, \alpha(p)) \\
&= \mu_2(\gamma([q]), x, \gamma([p])).
\end{aligned}$$

Therefore γ is a homomorphism.

Let $\gamma([p]), \gamma([q]) \in Q_2$ such that $\gamma([p]) = \gamma([q])$. Then $\alpha(q) = \alpha(p)$, i.e., $(q, p) \in SKer\,\alpha$. Hence $[p] = [q]$, so γ is one-one.

Let $q_2 \in Q_2$. As α is onto, there exists $q \in Q_1$ such that $q_2 = \alpha(q)$. Since $\alpha(q) = \gamma([q])$, there exists $[q] \in Q_1/(SKer\,\alpha)$ such that $q_2 \in \gamma([q])$. Hence γ is onto. Thus γ is an isomorphism. Furthermore, as γ is one-one, γ is a switching isomorphism. □

Definition 3.3.11. Let (Q_1, S_1, ρ_1) and (Q_2, S_2, ρ_2) be two ftss. Let $f : Q_1 \to Q_2$ and $g : S_1 \to S_2$ be mappings. A pair (f, g) is called a *switching homomorphism* from (Q_1, S_1, ρ_1) to (Q_2, S_2, ρ_2) if (f, g) is a homomorphism of ftss, and

$$\rho_2(f(q), g(x), f(p)) = \vee\{\rho_1(s, x, t) | s, t \in Q_1, f(t) = f(p), f(s) = f(q)\}$$

for $\forall q, p \in Q_1$ and $\forall x \in S_1$.

A switching homomorphism $(f, g) : (Q_1, S_1, \rho_1) \to (Q_2, S_2, \rho_2)$ is called a *switching isomorphism* if f and g are both one-one and onto.

Proposition 3.3.12. Let $M_1 = (Q_1, X_1, \mu_1)$ be a ffssm, and $M_2 = (Q_2, X_2, \mu_2)$ a ffsm. Let $(\alpha, \beta) : M_1 \to M_2$ be a switching homomorphism with α one-one and onto , and β onto. Then there exists a switching homomorphism

$(f_\alpha, g_\beta) : FTS(M_1) \to FTS(M_2)$ and $FTS(M_2)$ is a faithful fsts.

Proof. Let $FTS(M_1) = (Q_1, E(M_1), \rho_1)$ and $FTS(M_2) = (Q_2, E(M_2), \rho_2)$. Define $f_\alpha : Q_1 \to Q_2$ by $f_\alpha(q) = \alpha(q)$ for $\forall q \in Q_1$. Define $g_\beta : E(M_1) \to E(M_2)$ by $g_\beta([x]) = [\beta^*(x)]$ for $\forall [x] \in E(M_1)$.

Let $[x], [y] \in E(M_1)$ such that $[x] = [y]$. Then $\mu_1^*(q, x, p) = \mu_1^*(q, y, p)$ for $q, p \in Q_1$. Since α is one-one and onto, (α, β) is a switching homomorphism, by Theorem 3.1.10, then

$$\begin{aligned} \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) &= \mu_1^*(q, x, p) \\ &= \mu_1^*(q, y, p) \\ &= \mu_2^*(\alpha(q), \beta^*(y), \alpha(p)) \end{aligned}$$

for $\forall q, p \in Q_1$ and $\forall x \in X_1^*$. As α is onto,

$$\mu_2^*(q_2, \beta^*(x), p_2) = \mu_2^*(q_2, \beta^*(y), p_2)$$

for $\forall q_2, p_2 \in Q_2$. Hence $[\beta^*(x)] = [\beta^*(y)]$.

Let $[x], [y] \in E(M_1)$. By Lemma 3.1.8,

$$\begin{aligned} g_\beta([x] * [y]) &= g_\beta([xy]) \\ &= [\beta^*(xy)] \\ &= [\beta^*(x)\beta^*(y)] \\ &= [\beta^*(x)] * [\beta^*(y)] \\ &= g_\beta([x]) * g_\beta([y]). \end{aligned}$$

Let $[\lambda] \in E(M_1)$. Then

$$g_\beta([\lambda]) = [\beta^*(\lambda)] = [\lambda].$$

Let $[x] \in E(M_1)$ and $q, p \in Q_1$. By Theorem 3.1.10, then

$$\begin{aligned} \rho_1(q, [x], p) &= \mu_1^*(q, x, p) \\ &= \mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) \\ &= \rho_2(f_\alpha(q), g_\beta([x]), f_\alpha(p)). \end{aligned}$$

As α is one-one,

$$\begin{aligned} \rho_2(f_\alpha(q), g_\beta([x]), f_\alpha(p)) &= \vee\{\rho_1(s, [x], t) | s, t, \in Q_1, f_\alpha(t) = f_\alpha(p), f_\alpha(s) = f_\alpha(q)\} \\ &= \rho_1(q, [x], p). \end{aligned}$$

Hence (f_α, g_β) is a switching homomorphism.

Furthermore, by Proposition 3.3.3, M_2 is a ffssm. By Proposition 3.2.6, $FTS(M_2)$ is a faithful fsts. □

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