

On relations between generalized metric spaces and
hyperspaces

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Chapter 1

On generalized metric spaces and open problems

1.1 On Fréchet M_3 -spaces

We introduce property $(*)$ into M_3 -spaces and show that every closed subset of an M_3 -space with this property has a closure-preserving open neighborhood base in X , and consequently such space is an M_1 -space. This answers positively to the problem posed by Tamano whether Fréchet M_3 -spaces are M_1 .

1.1.1 Introduction.

Ceder defined class of M_i -spaces for $i=1,2,3$ and proved that $M_1 \rightarrow M_2 \rightarrow M_3$, [Ced61]. He asked there whether the reverses hold. Gruenhage [Gru76] and Junnila [Jun78] independently proved $M_2=M_3$. Borges renamed M_3 -spaces stratifiable ones and studied their properties [Bor66]. Thus, the problem whether M_3 implies M_1 remains unsolved and it is one of the most outstanding open problems in general topology.

For this problem, we have the partial answers one of which is due to Tamano [Tam89], where he proved that every Baire, Fréchet, M_3 -space is M_1 . The proof is the modification of Ito's discussion in [Ito85], which also gives the partial answer to this problem.

In this section, we define property $(*)$ which is weaker than Fréchet property, and prove that every M_3 -space with property $(*)$ is M_1 ; more strongly, every closed

subset of it has a closure-preserving open neighborhood base in it. This gives the positive answer to the problem whether every Fréchet M_3 -space is M_1 posed by Tamano [Tam89, Problem 3.6].

Throughout this section, we assume that all spaces are M_3 -spaces. Letter \mathbb{N} always denotes the set of natural numbers. For a space X , we denote the topology of X by $\tau(X)$. For any subset A of X , ∂A denotes the boundary of A in X . A family \mathcal{W} of subsets of X is called a *(closed, open) neighborhood base of a subset A of X in X* if \mathcal{W} consists of (closed, open, respectively) neighborhoods of A in X and if $A \subset U \in \tau(X)$, then $A \subset W \subset U$ for some $W \in \mathcal{W}$. For brevity, let the capital letter CP stand for “closure-preserving”.

The definitions of M_i -spaces and others are referred to the original papers [Ced61, Bor66] and their summation [Gru84]. We state the well-known properties of M_3 -spaces:

Fact 1.1.1 ([Ced61, Lemma 7.3]). Each closed subset of a space X has a CP closed neighborhood base in X .

Fact 1.1.2 ([Gru84, Theorem 5.16]). A space X has a monotonical normality operator D which assigns an open subset $D(H, K)$ to each pair of disjoint closed subsets of X such that

$$(i) \quad H \subset D(H, K) \subset D(H, K)^- \subset X \setminus K;$$

(ii) if $H \subset H'$ and $K' \subset K$, then

$$D(H, K) \subset D(H', K').$$

Fact 1.1.3 ([Gru84, Theorem 5.27]). A space X has the stratification

$S : \{\text{closed subsets of } X\} \times \mathbb{N} \rightarrow \tau(X)$ satisfies the following:

$$(i) \quad \text{For each closed } H, H = \bigcap \{S(H, n) \mid n \in \mathbb{N}\} = \bigcap \{S(H, n)^- \mid n \in \mathbb{N}\};$$

(ii) if $H \subset H'$, then $S(H, n) \subset S(H', n)$ for each n ;

(iii) for each closed subset F of X and $n \in \mathbb{N}$, $S(F, n+1)^- \subset S(F, n)$.

(This is not included in the original definition, but the modification is easy.)

Fact 1.1.4 ([SN68]). Let \mathcal{B} be a CP family of closed subsets of a space X . Then there exists a set $\{\mathcal{F}, \mathcal{V}\}$ of families of X satisfying the following:

(i) $\mathcal{F} = \bigcup\{\mathcal{F}_n | n \in \mathbb{N}\}$ is a cover of X , where each \mathcal{F}_n is a discrete family of closed subsets of X ;

(ii) $\mathcal{V} = \{V(F) | F \in \mathcal{F}\}$ is an open cover of X such that $F \subset V(F)$ for each $F \in \mathcal{F}$ and for each n , $\{V(F) | F \in \mathcal{F}_n\}$ discrete in X ;

(iii) for each $F \in \mathcal{F}$ and $B \in \mathcal{B}$, $F \cap B \neq \emptyset$ if and only if $F \subset B$, and if $F \cap B = \emptyset$, then $V(F) \cap B = \emptyset$.

We call \mathcal{F} the *mosaic on* \mathcal{B} and \mathcal{V} the *frill of* \mathcal{F} in X .

The next fact follows from Lemma 1.1.7 later as the special case:

Fact 1.1.5. Let \mathcal{B} be a CP family of closed subsets of a space X . Then there exists a set $\{\mathcal{W}(B) | B \in \mathcal{B}\}$ of families of X satisfying the following:

(i) $\bigcup\{\mathcal{W}(B) | B \in \mathcal{B}\}$ is a CP family of closed subsets of X ;

(ii) each $\mathcal{W}(B)$ is a closed neighborhood base of B in X .

1.1.2 Lemmas.

Lemma 1.1.6. Let M be a closed subset of a space X . Then there exists a mapping $T : \tau(X) \rightarrow \tau(X)$ satisfying the following:

(i) For each $O \in \tau(X)$, $T(O) \cap M = O \cap M$ and $T(O)^- \cap (X \setminus M) \subset O$;

(ii) if $O_1, O_2 \in \tau(X)$ and $O_1 \subset O_2$, then $T(O_1) \subset T(O_2)$.

Proof. Let D, S be the monotonical normality operator, the stratification of X , respectively. For each $O \in \tau(X)$, define

$$T(O) = \bigcup\{D(M \setminus S(X \setminus O, n), X \setminus (O \cap S(M, n))) | n \in \mathbb{N}\}.$$

Then it is easily checked that $\{T(O)|O \in \tau(X)\}$ satisfies the conditions (i) and (ii). \square

We call T the L_1 -operator with respect to M in X .

Lemma 1.1.7. *Let M be a closed subset of a space X . Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a set $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ of families of X satisfying the following:*

- (i) $\bigcup\{\mathcal{W}(B)|B \in \mathcal{B}\}$ is a CP family of closed subsets of X ;
- (ii) for each $B \in \mathcal{B}$, $\mathcal{W}(B)|M = \{B \cap M\}$ and $\mathcal{W}(B)|(X \setminus M)$ is a closed neighborhood base of $B \cap (X \setminus M)$ in $X \setminus M$.

Proof. Let \mathcal{F}' be the mosaic on $\mathcal{B} \cup \{M\}$ and \mathcal{F} the subfamily such that

$$\mathcal{F} = \{F \in \mathcal{F}' | F \cap M = \emptyset\}.$$

Note that \mathcal{F} is the mosaic on $\mathcal{B}|(X \setminus M)$. Let $\mathcal{F} = \bigcup\{\mathcal{F}_n | n \in \mathbb{N}\}$, where each \mathcal{F}_n is a discrete family of closed subsets of X . Let $\{V(F)|F \in \mathcal{F}'\}$ be the frill of \mathcal{F}' in X such that for each $F \in \mathcal{F}_n$, $n \in \mathbb{N}$,

$$F \subset V(F) \subset S(F, n).$$

By Fact 1.1.1, there exists a CP closed neighborhood base $\mathcal{B}(F)$ of F in X such that $\bigcup\mathcal{B}(F) \subset V(F)$. To construct $\mathcal{W}(B)$, $B \in \mathcal{B}$, let

$$\mathcal{F}(B) = \{F \in \mathcal{F} | F \subset B\}$$

and

$$\Delta(B) = \left\{ \delta = \langle B(F) \rangle \in \prod\{\mathcal{B}(F) | F \in \mathcal{F}(B)\} \mid \right. \\ \left. W(\delta) = \bigcup\{B(F) | F \in \mathcal{F}(B)\} \text{ satisfies } W(\delta) \cap T(X \setminus B)^- = \emptyset \right\}$$

where T is the L_1 -operator with respect to M in X . Note that $W(\delta)$ is a neighborhood of $B \cap (X \setminus M)$ in $X \setminus M$ for each $\delta \in \Delta(B)$. If we define

$$\mathcal{W}(B) = \{B \cup W(\delta) \mid \delta \in \Delta(B)\}, \quad B \in \mathcal{B},$$

then we can show that $\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ has the required properties. (ii) is easily checked. To see (i), let $\mathcal{B}_0 \subset \mathcal{B}$ and let $\mathcal{W}_0(B) \subset \mathcal{W}(B)$ for each $B \in \mathcal{B}_0$. Suppose $p \notin \bigcup \{W^- \mid W \in \bigcup \{\mathcal{W}_0(B) \mid B \in \mathcal{B}_0\}\}$. Take $n \in \mathbb{N}$ such that $p \notin S(\bigcup \mathcal{B}_0, n)^-$. Note that $\bigcup \mathcal{B}(F) \subset S(\bigcup \mathcal{B}_0, n)$ for each $F \in (\bigcup \{\mathcal{F}(B) \mid B \in \mathcal{B}_0\}) \cap (\bigcup \{\mathcal{F}_k \mid k \geq n\})$. On the other hand, it is easily observed that $\bigcup \{\mathcal{B}(F) \mid F \in (\bigcup \{\mathcal{F}(B) \mid B \in \mathcal{B}_0\}) \cap (\bigcup \{\mathcal{F}_k \mid k < n\})\}$ is CP in X . Thus, we can easily find an open neighborhood O of p in X such that $O \cap W = \emptyset$ for each $W \in \bigcup \{\mathcal{W}_0(B) \mid B \in \mathcal{B}_0\}$. \square

We call $\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ the L_2 -extension of \mathcal{B} with respect to M in X .

From here, we assume that a space X has property $(*)$, which we define next.

Definition 1.1.8. A space X has *property $(*)$* if every $U \in \tau(X)$ satisfies the following condition:

$(*)$ If $p \in \partial U$, then there exists a CP family \mathcal{G} of closed subsets of U^- such that for each $G \in \mathcal{G}$, $(G \setminus \{p\})^- = G$, $G \cap \partial U = \{p\}$ and if $p \in O \in \tau(X)$, then $p \in G \subset O$ for some $G \in \mathcal{G}$.

Lemma 1.1.9. Let $O \in \tau(Y)$, $X = O^-$ and $M = \partial O$, where Y is a space with property $(*)$. Let \mathcal{B} be a CP family of closed subsets of M . Then there exists a set $\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ of families of closed subsets of X satisfying the following:

(i) $\bigcup \{\mathcal{W}(B) \mid B \in \mathcal{B}\}$ is CP in X ;

(ii) for each $B \in \mathcal{B}$, $\mathcal{W}(B) \mid M = \{B\}$ and for each $W \in \mathcal{W}(B)$,

$$(\text{Int } W)^- \cap M = B;$$

(iii) if $B \subset U \in \tau(X)$, $B \in \mathcal{B}$, then there exists $W \in \mathcal{W}(B)$ such that $B \subset W \subset U$.

Proof. By Fact 1.1.4, there exists a σ -discrete closed subset D of M such that for each $B \in \mathcal{B}$, $(D \cap B)^- = B$. Let $D = \bigcup\{D_n | n \in \mathbb{N}\}$, where each D_n is discrete closed in M . For each n , we take a discrete family $\{V(p) | p \in D_n\}$ of open subsets of X such that $p \in V(p)$ for each $p \in D_n$. By property (*), for each $p \in D$, there exists a CP family $\mathcal{G}(p)$ of closed subsets of X satisfying the following:

- (1) $\bigcup \mathcal{G}(p) \subset S(\{p\}, n) \cap V(p)$ for each $p \in D_n$, $n \in \mathbb{N}$;
- (2) for each $G \in \mathcal{G}(p)$, $G \cap M = \{p\}$ and $(G \setminus \{p\})^- = G$;
- (3) if $p \in U \in \tau(X)$, then $p \in G \subset U$ for some $G \in \mathcal{G}(p)$.

For each $B \in \mathcal{B}$, define $\mathcal{W}'(B)$ as follows:

$$\Delta(B) = \left\{ \delta = \langle G(p) \rangle \in \prod \{ \mathcal{G}(p) | p \in D \cap B \} \right\}, \quad \mathcal{W}'(B) = \{ B \cup G(\delta) | \delta \in \Delta(B) \},$$

where

$$G(\delta) = \bigcup \{ G(p) | p \in D \cap B \}$$

for $\delta = \langle G(p) \rangle \in \Delta(B)$. Then we show that $\{\mathcal{W}'(B) | B \in \mathcal{B}\}$ has the following properties:

- (4) $\bigcup \{ \mathcal{W}'(B) | B \in \mathcal{B} \}$ is a CP family of closed subsets of X ;
- (5) for each $B \in \mathcal{B}$, $\mathcal{W}'(B) | M = \{B\}$ and
for each $W \in \mathcal{W}'(B)$, $(W \setminus M)^- \cap M = \{B\}$;
- (6) if $B \subset U \in \tau(X)$, $B \in \mathcal{B}$, then $B \subset W \subset U$ for some $W \in \mathcal{W}'(B)$.

To see (4), let $\mathcal{B}_0 \subset \mathcal{B}$ and $\mathcal{W}_0(B) \subset \mathcal{W}'(B)$ for each $B \in \mathcal{B}_0$. Suppose $p \notin \bigcup \{ W^- | W \in \mathcal{W}_0 \}$, where $\mathcal{W}_0 = \bigcup \{ \mathcal{W}_0(B) | B \in \mathcal{B}_0 \}$. Take $n \in \mathbb{N}$ such that $p \notin S(\bigcup \mathcal{B}_0, n)^-$. By (1), it is easily observed that

$$\bigcup \left\{ \mathcal{G}(p) \mid p \in \bigcup \{ D_k | k < n \} \right\}$$

is a CP family of closed subsets of X . Thus we can easily find an open neighborhood O of p in X such that $O \cap W = \emptyset$ for each $W \in \mathcal{W}_0$. Hence $\bigcup\{\mathcal{W}'(B)|B \in \mathcal{B}\}$ is CP in X . If we apply the same discussion as above to each $W \in \bigcup\{\mathcal{W}'(B)|B \in \mathcal{B}\}$, then W is shown to be closed in X . (5) follows easily from (2) and the closedness of W above. (6) follows easily from (3). By Lemma 1.1.7, we can take the L_2 -extension $\{\mathcal{W}''(W')|W' \in \mathcal{W}'\}$ of

$$\mathcal{W}' = \bigcup\{\mathcal{W}'(B)|B \in \mathcal{B}\}$$

with respect to M in X . For each $B \in \mathcal{B}$, we define $\mathcal{W}(B)$ as follows:

$$\mathcal{W}(B) = \bigcup\{\mathcal{W}''(W')|W' \in \mathcal{W}'(B)\}.$$

Then it is easy from (4), (5) and (6) to see that $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ has the required properties. \square

We call $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ the L_3 -extension of \mathcal{B} to X in Y with respect to O .

Lemma 1.1.10. *Let O, M, X, Y be the same as in Lemma 1.1.9. Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a set $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ of families of closed subsets of X satisfying the following:*

- (i) $\bigcup\{\mathcal{W}(B)|B \in \mathcal{B}\}$ is CP in X ;
- (ii) for each $B \in \mathcal{B}$, $\mathcal{W}(B)|M = \{B \cap M\}$ and for each $W \in \mathcal{W}(B)$,

$$(\text{Int } W)^- \cap M = B \cap M;$$

- (iii) if $B \subset U \in \tau(X)$, $B \in \mathcal{B}$, then $B \subset W \subset U$ for some $W \in \mathcal{W}(B)$.

Proof. Let $\{\mathcal{W}'(B \cap M)|B \in \mathcal{B}\}$ be the L_3 -extension of $\mathcal{B}|M$ to X in Y with respect to O . For each $B \in \mathcal{B}$, let

$$\mathcal{W}(B) = \{B \cup W|W \in \mathcal{W}'(B \cap M)\}.$$

Then it is easy to see that $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ has the required properties. \square

We call $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ the L_4 -extension of \mathcal{B} to X in Y with respect to O .

Lemma 1.1.11. *Let $\{X(n)|n \in \mathbb{N}\}$ be a disjoint cover of a space X with property (*) such that for each n*

$$\bigcup\{X(t)|t \leq n\} \text{ is closed in } X \text{ and } \left(\bigcup\{X(t)|t > n\}\right)^- \supset X(n).$$

(For brevity, we call such $\{X(n)\}$ the special partition of X .) Let \mathcal{B} be a CP family of closed subsets X . Then there exists a set $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ of families of X satisfying the following:

- (i) $\bigcup\{\mathcal{W}(B)|B \in \mathcal{B}\}$ is a CP family of regular closed subsets of X ;
- (ii) for each $B \in \mathcal{B}$, if $B \subset U \in \tau(X)$, then there exists $W \in \mathcal{W}(B)$ such that $B \subset \text{Int } W \subset W \subset U$.

Proof. By Fact 1.1.5, there exists a set $\{\mathcal{W}(B; 1)|B \in \mathcal{B}\}$ of families of X satisfying the following:

- (1) $\mathcal{W}(1) = \bigcup\{\mathcal{W}(B; 1)|B \in \mathcal{B}\}$ is a CP family of closed subsets of X and each $\mathcal{W}(B; 1)$ is a closed neighborhood base of B in X .

By Lemma 1.1.10, there exists the L_4 -extension $\{\mathcal{W}(W_1; 2)|W_1 \in \mathcal{W}(1)\}$ of $\mathcal{W}(1)$ to X in X with respect to $X \setminus X(1)$ satisfying the following:

- (2) $\mathcal{W}(2) = \bigcup\{\mathcal{W}(W_1; 2)|W_1 \in \mathcal{W}(1)\}$ is a CP family of closed subsets of X ;

- (3) for each $W_2 \in \mathcal{W}(W_1; 2)$, $W_1 \in \mathcal{W}(1)$,

$$W_2 \cap X(1) = W_1 \cap X(1) \text{ and } (\text{Int } W_2)^- \cap X(1) = W_1 \cap X(1);$$

- (4) if $W_1 \subset U \in \tau(X)$, $W_1 \in \mathcal{W}(1)$, then

$$W_1 \subset W_2 \subset U \text{ for some } W_2 \in \mathcal{W}(W_1; 2).$$

Assume that we have constructed the family

$$\mathcal{W}(n) = \bigcup \{\mathcal{W}(W; n) | W \in \mathcal{W}(n-1)\}.$$

It is easily checked that property (*) is hereditary with respect to any open subspaces of X . So, we can apply Lemma 1.1.10 to the family $\mathcal{W}(n)|Y$, where $Y = \bigcup \{X(t) | t \geq n\}$, in the open subspace Y to obtain the L_4 -extension

$$\{\mathcal{W}'(W \cap Y) | W \in \mathcal{W}(n)\}$$

of $\mathcal{W}(n)|Y$ to Y in Y with respect to $Y \setminus X(n)$, which satisfies the following:

- (5) $\bigcup \{\mathcal{W}'(W \cap Y) | W \in \mathcal{W}(n)\}$ is a CP family of closed subsets of Y ;
- (6) for each $W \in \mathcal{W}(n)$, $\mathcal{W}'(W \cap Y)|X(n) = \{W \cap X(n)\}$ and
for each $W' \in \mathcal{W}'(W \cap Y)$, $(\text{Int } W')^- \cap X(n) = W \cap X(n)$;
- (7) if $W \cap Y \subset O \in \tau(X)$, then $W \cap Y \subset W' \subset O$ for some $W' \in \mathcal{W}'(W \cap Y)$.

For each $W \in \mathcal{W}(n)$, let

$$\mathcal{W}(W; n+1) = \{W \cup W' | W' \in \mathcal{W}'(W \cap Y) \text{ and } W' \cap T_{n-1}(X \setminus W)^- = \emptyset\},$$

where T_{n-1} is the L_1 -operator with respect to $\bigcup \{X(t) | t \leq n-1\}$ in X , and let

$$\mathcal{W}(n+1) = \bigcup \{\mathcal{W}(W; n+1) | W \in \mathcal{W}(n)\}.$$

Then by the properties (5), (6) and (7), it is easy to see that $\mathcal{W}(n+1)$ has the following properties:

- (8) $\mathcal{W}(n+1)$ is a CP family of closed subsets of X ;
- (9) for each $W_2 \in \mathcal{W}(W_1; n+1)$, $W_1 \in \mathcal{W}(n)$,
 $W_2 \cap \left(\bigcup \{X(t) | t \leq n\} \right) = W_1 \cap \left(\bigcup \{X(t) | t \leq n\} \right)$ and
 $(\text{Int } W_2)^- \cap X(n) = W_2 \cap X(n)$;

(10) if $W_1 \cap \left(\bigcup \{X(t) | t \geq n\} \right) \subset O \in \tau(X)$, $W_1 \in \mathcal{W}(n)$, then

there exists $W_2 \in \mathcal{W}(W_1; n+1)$ such that $W_1 \subset W_2$ and

$$W_2 \cap \left(\bigcup \{X(t) | t \geq n\} \right) \subset O \text{ for some } W_2 \in \mathcal{W}(W_1; n+1).$$

Let T_i be the L_1 -operator with respect to $\bigcup \{X(t) | t \leq i\}$ in X . With these preliminaries, we construct the required families $\{\mathcal{W}(B) | B \in \mathcal{B}\}$. Let $B \in \mathcal{B}$ be fixed for a while. Let $\Delta(B)$ be the totality of

$$\delta = \langle W(n) \rangle \in \prod \{\mathcal{W}(n) | n \in \mathbb{N}\}$$

which satisfies the following two conditions:

(11) $W(1) \in \mathcal{W}(B; 1)$ and $W(n+1) \in \mathcal{W}(W(n); n+1)$ for each n ;

(12) $W(n+1) \cap \left(\bigcup \{X(t) | t \geq n\} \right) \cap (T_1(X \setminus W(2))^- \cup \dots \cup T_{n-1}(X \setminus W(n))^-) = \emptyset$ for each n .

For each $\delta = \langle W(n) \rangle \in \Delta(B)$, let $W(\delta) = \bigcup \{W(n) | n \in \mathbb{N}\}$. Then we can show that $\mathcal{W}(B) = \{W(\delta) | \delta \in \Delta(B)\}$ has the following property:

(13) Each $W(\delta) \in \mathcal{W}(B)$ is regular closed in X .

To see that $W(\delta)$ is closed in X , let $p \in (X \setminus W(\delta)) \cap X(n)$; then $p \in T_n(X \setminus W(n+1))$, which is an open neighborhood of p in X missing $W(\delta)$ by (12). On the other hand, by (3), (9) and the definition of $\delta \in \Delta(B)$, we have

$$W(\delta) = \left(\bigcup \{\text{Int } W(n) | n \in \mathbb{N}\} \right)^-.$$

By the similar way to the above, the following is true:

(14) $\bigcup \{\mathcal{W}(B) | B \in \mathcal{B}\}$ is CP in X .

Since obviously $\mathcal{W}(B)$ forms a neighborhood base of B in X , the proof is completed. □

Remark 1. Let us recall that M_3 -spaces are hereditary and property $(*)$ is open hereditary. Thus we can state the following: Let X , $\{X(n)\}$, \mathcal{B} be the same as above except for that X is an open subspace of a space Y with property $(*)$. Then there exists a set $\{\mathcal{W}(B)|B \in \mathcal{B}\}$ of families of X satisfying the same (i) and (ii) as above.

Lemma 1.1.12. *Let Z be an open subset of a space X with property $(*)$. Let $\mathcal{B} = \{B(\lambda)|\lambda \in \Lambda\}$ be a CP family of closed subsets of the subspace Z^- and $\{O(\lambda)|\lambda \in \Lambda\}$ a family of open subsets of Z^- such that $B(\lambda) \subset O(\lambda)$ for each $\lambda \in \Lambda$. Let \mathcal{V} be a disjoint open cover of the subspace Z . Then there exists a family $\{W(\lambda)|\lambda \in \Lambda\}$ of Z^- satisfying the following:*

- (i) $\{W(\lambda)|\lambda \in \Lambda\}$ is a CP family of regular closed subsets of Z^- ;
- (ii) for each $\lambda \in \Lambda$, $W(\lambda) \cap \partial Z = B(\lambda) \cap \partial Z$ and

$$B(\lambda) \subset W(\lambda) \subset \text{St}(O(\lambda), \mathcal{V}).$$

Proof. By Lemma 1.1.10, there exists the L_4 -extension $\{\mathcal{W}(B(\lambda))|\lambda \in \Lambda\}$ of \mathcal{B} to Z^- in X with respect to Z .

For each $\lambda \in \Lambda$, we take $W_1(\lambda) \in \mathcal{W}(B(\lambda))$ such that $B(\lambda) \subset W_1(\lambda) \subset O(\lambda)$. Note that $\{W_1(\lambda)|\lambda \in \Lambda\}$ is a CP family of closed subsets of Z^- such that $(\text{Int } W_1(\lambda))^- \cap \partial Z = B(\lambda) \cap \partial Z$ for each $\lambda \in \Lambda$. Thus there exists a σ -discrete closed subset D of Z^- such that for each $\lambda \in \Lambda$

$$W_1(\lambda) = (W_1(\lambda) \cap D \cap Z)^-.$$

Let $D = \bigcup \{D_n | n \in \mathbb{N}\}$, where each D_n is a discrete closed subset of Z^- . For each $p \in D_n \cap Z$, $n \in \mathbb{N}$, we choose an open neighborhood $O(p)$ of p in X such that

$$(1) \quad p \in O(p) \subset O(p)^- \subset S(\{p\}, n) \cap V(p),$$

where $V(p)$ is the unique member of \mathcal{V} with $p \in V(p)$.

Moreover, we can assume that

$$(2) \quad \{O(p) \mid p \in D_n \cap Z\} \text{ is discrete in } Z^-.$$

For each $\lambda \in \Lambda$, we define $W(\lambda)$ as follows:

$$W(\lambda) = W_1(\lambda) \cup \left(\bigcup \{O(p)^- \mid p \in D \cap W_1(\lambda) \cap Z\} \right).$$

Then it is easy to see that $\{W(\lambda) \mid \lambda \in \Lambda\}$ has the required properties. \square

1.1.3 The main result.

Theorem 1.1.13. *If X is an M_3 -space with property $(*)$, then every closed subset of X has CP open neighborhood base in X , hence, X is an M_1 -space.*

Proof. Let M be any closed subset of X . It suffices to show that there exists a CP neighborhood base of M in X , consisting of regular closed neighborhoods of M in X [BL74, Theorem 2.6].

By Fact 1.1.1, there exists a CP closed neighborhood base \mathcal{B} of M in X . By Fact 1.1.4, \mathcal{B} has the mosaic \mathcal{F} on it such that $\mathcal{F} = \bigcup \{\mathcal{F}(n) \mid n \in \mathbb{N}\}$, where each $\mathcal{F}(n)$ is a discrete family of closed subsets of X . For each n , let

$$\begin{aligned} X'(1) &= \bigcup \mathcal{F}(1), \\ X'(n+1) &= \bigcup \mathcal{F}(n+1) \setminus (X'(1) \cup \dots \cup X'(n)), \\ Z(n) &= \text{Int } X'(n), \quad Z = \bigcup \{Z(n) \mid n \in \mathbb{N}\}, \\ Y &= X \setminus Z^-, \quad X(n) = X'(n) \cap Y. \end{aligned}$$

Then we can show that $\{X(n) \mid n \in \mathbb{N}\}$ is a special partition of an open subspace Y . Since for each $n \in \mathbb{N}$, $\bigcup \mathcal{F}(n)$ is closed in X , $\bigcup \{X(k) \mid k \leq n\}$ is closed in Y . We can easily check $\text{Cl}_Y(\bigcup \{X(k) \mid k > n\}) \supset X(n)$ for each n . So, we can apply Remark 1 to a CP family $\mathcal{B} \mid Y$ of closed subsets of Y . Then there exists a set $\{\mathcal{W}(B \cap Y) \mid B \in \mathcal{B}\}$ of families of Y satisfying the following:

- (1) $\bigcup \{\mathcal{W}(B \cap Y) \mid B \in \mathcal{B}\}$ is a CP family of regular closed subsets of Y
such that $\left(\bigcup \mathcal{W}(B \cap Y) \right) \cap T(X \setminus B)^- = \emptyset,$

where T is the L_1 -operator with respect to Z^- in X .

(2) For each $B \in \mathcal{B}$, if $B \cap Y \subset U \in \tau(X)$, then

there exists $W \in \mathcal{W}(B \cap Y)$ such that $B \cap Y \subset \text{Int } W \subset W \subset U$.

Let Δ be the totality of pairs $\delta = (B_1, B_2)$ of members of \mathcal{B} such that $B_1 \subset \text{Int } B_2$. By Lemma 1.1.12, for each $\delta = (B_1, B_2) \in \Delta$, there exists a subset $W(\delta)$ of the subspace Z^- satisfying the following:

(3) $\{W(\delta) \mid \delta \in \Delta\}$ is CP in Z^- ;

(4) for each $\delta = (B_1, B_2) \in \Delta$, $W(\delta)$ is a regular closed subset of Z^- satisfying

$$W(\delta) \cap \partial Z = B_1 \cap \partial Z \quad \text{and} \quad B_1 \cap Z^- \subset W(\delta) \subset \text{St}(\text{Int } B_2, \mathcal{V}),$$

where $\mathcal{V} = \mathcal{F}|Z$ is a disjoint open cover of Z . Note that the last relation above implies $W(\delta) \subset B_2$. For each pair $\delta = (B_1, B_2) \in \Delta$, define

$$\mathcal{W}(\delta) = \{W \cup W(\delta) \mid W \in \mathcal{W}(B_1 \cap Y)\}.$$

By virtue of (1), (2), (3) and (4), $\{\mathcal{W}(\delta) \mid \delta \in \Delta\}$ has the following properties:

(5) $\bigcup \{\mathcal{W}(\delta) \mid \delta \in \Delta\}$ is a CP family of regular closed neighborhoods of M in X ;

(6) if $M \subset O \in \tau(X)$, then

there exist $\delta \in \Delta$ and $W \in \mathcal{W}(\delta)$ such that $M \subset \text{Int } W \subset W \subset O$.

This completes the proof. □

Since it is easily checked that Fréchet spaces has property (*), the theorem gives the positive answer to Tamano's problem [Tam89, Problem 3.6].

Corollary 1.1.14. *Every Fréchet M_3 -space is M_1 .*

Here, we give two comments on property (*).

Remark 2. Property (*) is equivalent to property (**) if the space X is an M_3 -space:
(**) For any $U \in \tau(X)$ and any $p \in \partial U$, there exists a closed set F of X such that $F \setminus \{p\} \subset U$, and $(F \setminus \{p\})^- = F$.

Indeed, clearly (*) implies (**). To show the converse, assume (**). Take U , p and F as above. Since X is an M_3 -space, p has a CP closed neighborhood base \mathcal{H} . Then $\mathcal{G} = \mathcal{H}|F$ satisfies (*).

Example 1.1.15. There exists an M_3 -space which does not satisfy (*).

Construction. As the space X , let

$$X = (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \setminus (\mathbb{N} \times \{\infty\}).$$

All points in $\mathbb{N} \times \mathbb{N}$ are isolated. For each $n \in \mathbb{N}$, the point (∞, n) has the set of all $(\{\infty\} \cup \{i | i \geq m\}) \times \{n\}$, for $m \in \mathbb{N}$, for a neighborhood base. The neighborhood base of (∞, ∞) consists of sets of the form $\{(\infty, \infty)\} \cup (\bigcup_{m \geq n} (\{\infty\} \cup \{i | i \geq f(m)\}) \times \{m\})$, for $n \in \mathbb{N}$, and $f : \mathbb{N} \rightarrow \mathbb{N}$. It is easy to check that X is an M_1 -space. Let $U = \mathbb{N} \times \mathbb{N}$ and $p = (\infty, \infty)$. Then $p \in \partial U$. Note that $U \cup \{p\}$ is Arens' space and there is no closed set F of X satisfying (**). Hence X does not satisfy (*). \square

Finally, we express our thanks to the referee for his comment. Especially we should say that the last example is due to his idea.

1.2 On the M_3 vs. M_1 problem

We introduce a technical property (P) and use it to show that every M_3 -space with (P) is an M_1 -space whose every closed subset has a closure-preserving open neighborhood base. We show that every M_3 k-space has property (P) and is, therefore, an M_1 -space.

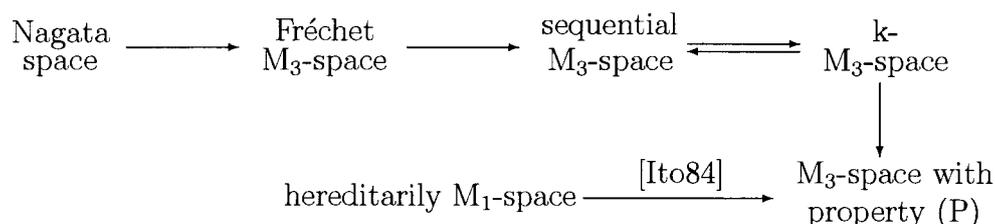
1.2.1 Introduction

In this section we use a new technical property to prove that any M_3 -space that is a k-space must be M_1 . Recall that the classes of M_1 , M_2 , and M_3 -spaces were introduced and studied by Ceder in [Ced61] and that, in the later paper [Bor66], Borges studied M_3 -spaces, re-naming them “stratifiable spaces”. It was always clear that $M_1 \Rightarrow M_2 \Rightarrow M_3$, and a classical problem asked which of those implications could be reversed. The best result to date is due to Gruenhage [Gru76] and Junnila [Jun78] independently who proved that $M_3 = M_2$. Whether $M_3 \Rightarrow M_1$ is still open.

The purpose of this section is to show that certain large classes of M_3 -spaces are M_1 . To do that we introduce a technical property called property (P) (defined in Definition 1.2.8 of Subsection 1.2.2) and show

- A) Every M_3 -space that is a k-space must have property (P) (Theorem 1.2.15);
- B) Every M_3 -space with property (P) is M_1 and has the additional property that each of its closed subsets has a closure-preserving open base (Theorem 1.2.14).

Together with other known results, this gives



(Recall that a Nagata space is a first-countable M_3 -space and recall the equivalence of being sequential and being k among M_3 -spaces because of [Gru84, Theorem 2.13].)

For general information and general properties of M_3 -spaces, see Gruenhage's survey paper [Gru84]. Since, as noted above, $M_3 = M_2$, we will use the fact that every M_3 -space has both a stratification and a σ -closure-preserving quasi-base. We denote by N the set of natural numbers. For a space X , let $\tau(X)$ denote the topology of X . For a subset A of X , let $\text{Int } A$, A^- , ∂A denote the interior, closure, boundary of A in X , respectively. A family \mathcal{W} of subsets of X is called a (closed, open) neighborhood base of A in X if \mathcal{W} consists of (closed, open, respectively) neighborhoods of A in X and if $A \subset U \in \tau(X)$, then $A \subset W \subset U$ for some $W \in \mathcal{W}$. For brevity, let "CP" stand for the term "closure-preserving".

1.2.2 Facts and Lemmas.

We state the well-known properties of M_3 -spaces or stratifiable spaces:

Fact 1.2.1 ([Ced61, Lemma 7.3]). Each closed subset of an M_3 -space X has a CP closed neighborhood base in X .

Fact 1.2.2 ([Gru84, Theorem 5.16]). A space X has a monotonical normality operator D which assigns an open subset $D(H, K)$ to each pair (H, K) of disjoint closed subsets of X such that

- (i) $H \subset D(H, K) \subset D(H, K)^- \subset X \setminus K$;
- (ii) if $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$.

Fact 1.2.3. A space X is an M_3 -space if and only if there exists a stratification $S : \{\text{closed subsets of } X\} \times N \rightarrow \tau(X)$ satisfying the following:

- (i) For each closed subset F of X ,

$$F = \bigcap \{S(F, n) : n \in N\} = \bigcap \{S(F, n)^- : n \in N\};$$

- (ii) if F, F' are closed in X and $F \subset F'$, then $S(F, n) \subset S(F', n)$ for each n ;

(iii) for each n and each F ,

$$S(F, n + 1)^- \subset S(F, n).$$

(Originally, the term stratification is used for S with (i) and (ii), but one with (i), (ii) and (iii) follows by a slight change.)

Fact 1.2.4 (Essentially in [SN68]). Let \mathcal{B} be a CP family of closed subsets of an M_3 -space X . Then there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of families of subsets of X satisfying the following:

- (i) \mathcal{F} is a star-finite, σ -discrete closed cover of X such that if $F \in \mathcal{F}$ and $B \in \mathcal{B}$, then $B \cap F \neq \emptyset$ if and only if $F \subset B$;
- (ii) $\mathcal{V} = \{V(F) : F \in \mathcal{F}\}$ is a point-finite, σ -discrete open cover of X such that $F \subset V(F)$ for each $F \in \mathcal{F}$ and such that if $F \cap B = \emptyset$, $F \in \mathcal{F}$, $B \in \mathcal{B}$, then $V(F) \cap B = \emptyset$.

Since our proof of Theorem 1.2.14 depends on the construction of $\langle \mathcal{F}, \mathcal{V} \rangle$, we sketch it roughly: Let $\mathcal{P} = \{P(\delta) : \delta \in \Delta\}$ be the partition of X by $\mathcal{B}' = \mathcal{B} \cup \{X\}$, that is, for each $\delta \in \Delta$,

$$P(\delta) = \bigcap \mathcal{B}(\delta) \setminus \bigcup (\mathcal{B}' \setminus \mathcal{B}(\delta)),$$

where $\mathcal{B}(\delta) \subset \mathcal{B}'$. For each $n \in N$, let

$$F(\delta, n) = P(\delta) \cap (S(T(\delta), n - 1)^- \setminus S(T(\delta), n)),$$

where $T(\delta) = \bigcup (\mathcal{B}' \setminus \mathcal{B}(\delta))$ and $S(\cdot, 0) = X$. Then it is easy to check that $\mathcal{F}(n) = \{F(\delta, n) : \delta \in \Delta\}$ is a discrete family of closed subsets of X and $\mathcal{F} = \bigcup \{\mathcal{F}(n) : n \in N\}$ is star-finite. Since X is paracompact, there exists a discrete open expansion

$$\mathcal{V}(n) = \{V(\delta, n) : \delta \in \Delta\}, \quad n \in N,$$

of $\mathcal{F}(n)$ such that

$$F(\delta, n) \subset V(\delta, n) \subset S(F(\delta, n), n) \cap (S(T(\delta), n - 2) \setminus S(T(\delta), n + 1)^-)$$

for each $\delta \in \Delta$, $n \in N$, where $S(\cdot, -1) = X$. Then it is easy to see that $\mathcal{V} = \bigcup\{\mathcal{V}(n) : n \in N\}$ is the required one. \square

We call \mathcal{F} the tile on \mathcal{B} in X and \mathcal{V} the frill of \mathcal{F} in X .

Fact 1.2.5 ([Ito85]). Let \mathcal{B} be a CP family of closed subsets of an M_3 -space X . Then there exists a σ -closed-discrete subset D of X such that for each $B \in \mathcal{B}$, $B = (D \cap B)^-$, where we call D a σ -closed-discrete subset of X if $D = \bigcup\{D_n : n \in N\}$ with each D_n discrete and closed in X .

Lemma 1.2.6. Let M be a closed subset of an M_3 -space X . Then there exists a mapping $T : \tau(X) \rightarrow \tau(X)$ satisfying the following:

(i) For each $O \in \tau(X)$, $T(O) \cap M = O \cap M$ and

$$T(O)^- \cap (X \setminus M) \subset O;$$

(ii) if $O_1, O_2 \in \tau(X)$ and $O_1 \subset O_2$, then $T(O_1) \subset T(O_2)$.

Proof. Let D and S be the monotonical normality operator and the stratification of X , respectively. For each $O \in \tau(X)$, define

$$T(O) = \bigcup\{D(M \setminus S(X \setminus O, n), X \setminus (O \cap S(M, n))) : n \in N\}.$$

Then it is easy to see that $\{T(O) : O \in \tau(X)\}$ satisfies the conditions (i) and (ii). \square

We call T the $L1$ -operator with respect to M in X .

Lemma 1.2.7. Let M be a closed subset of an M_3 -space X . Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of X satisfying the following:

(i) $\bigcup\{\mathcal{W}(B) : B \in \mathcal{B}\}$ is a CP family of closed subsets of X ;

(ii) for each $B \in \mathcal{B}$, $\mathcal{W}(B)|_M = \{B \cap M\}$ and $\mathcal{W}(B)|(X \setminus M)$ is a closed neighborhood base of $B \cap (X \setminus M)$ in $X \setminus M$.

Proof. Apply Fact 1.2.4 to the CP family $\mathcal{B} \cup \{M\}$ to find \mathcal{F}' , the tile on $\mathcal{B} \cup \{M\}$, and consider the subfamily defined by

$$\mathcal{F} = \{F \in \mathcal{F}' : F \cap M = \emptyset\}.$$

Note that \mathcal{F} is written as $\mathcal{F} = \bigcup\{\mathcal{F}(n) : n \in N\}$, where each $\mathcal{F}(n)$ is a discrete family of closed subsets of X . Let $\{V(F) : F \in \mathcal{F}'\}$ be the frill of \mathcal{F}' in X such that for each $F \in \mathcal{F}(n)$, $n \in N$,

$$F \subset V(F) \subset S(F, n).$$

By Fact 1.2.1, there exists a CP closed neighborhood base $\mathcal{B}(F)$ of F in X such that $\bigcup \mathcal{B}(F) \subset V(F)$. To construct $\mathcal{W}(B)$ for each $B \in \mathcal{B}$, let

$$\mathcal{F}(B) = \{F \in \mathcal{F} : F \subset B\}$$

and

$$\Delta(B) = \left\{ \delta = \langle B(F) \rangle \in \prod \{\mathcal{B}(F) : F \in \mathcal{F}(B)\} : \right.$$

$$W(\delta) = \bigcup \{B(F) : F \in \mathcal{F}(B)\} \text{ is a neighborhood of } B \cap (X \setminus M)$$

$$\left. \text{in } X \setminus M \text{ and } W(\delta) \cap T(X \setminus B)^- = \emptyset \right\},$$

where T is the L1-operator with respect to M in X . If we define

$$\mathcal{W}(B) = \{B \cap W(\delta) : \delta \in \Delta(B)\}, \quad B \in \mathcal{B},$$

then it is easy to see that $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ has the required properties. \square

We call $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ the *L2-extension of \mathcal{B} in X* .

We introduce property (P) as follows:

Definition 1.2.8. A space X has *property (P)* if every $U \in \tau(X)$ satisfies the following condition:

(*) If $p \in \partial U$, then there exists a CP family \mathcal{G} of closed subsets of U^- such that for each $G \in \mathcal{G}$, $(G \cap U)^- = G$ and if $p \in O \in \tau(X)$, then $p \in G \subset O$ for some $G \in \mathcal{G}$.

Lemma 1.2.9. *Let $O \in \tau(Y)$, $X = O^-$ and $M = \partial O$, where Y is an M_3 -space with property (P). Let \mathcal{B} be a CP family of closed subsets of M . Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of closed subsets of the subspace X satisfying the following:*

(i) $\bigcup\{\mathcal{W}(B) : B \in \mathcal{B}\}$ is CP in X ;

(ii) for each $W \in \mathcal{W}(B)$ and each $B \in \mathcal{B}$,

$$(\text{Int } W)^- \cap M = W \cap M;$$

(iii) if $B \subset U \in \tau(X)$, then there exists $W \in \mathcal{W}(B)$ such that $B \subset W \subset U$.

Proof. By Fact1.2.5, there exists a σ -closed-discrete subset D of M such that for each $B \in \mathcal{B}$, $(D \cap B)^- = B$. Let $D = \bigcup\{D_n : n \in N\}$, where each D_n is discrete and closed in M . For each n , we take a discrete family $\{V(p) : p \in D_n\}$ of open subsets of X such that $p \in V(p)$ for each $p \in D_n$. By property (P), for each $p \in D$, there exists a CP family $\mathcal{G}(p)$ of closed subsets of X satisfying the following:

(1) $\bigcup\mathcal{G}(p) \subset S(\{p\}, n) \cap V(p)$ for each $p \in D_n$, $n \in N$;

(2) for each $G \in \mathcal{G}(p)$, $(G \setminus M)^- = G$ and

if $p \in O \in \tau(X)$, then there exists $G \in \mathcal{G}(p)$ such that $p \in G \subset O$.

For each $B \in \mathcal{B}$, define $\mathcal{W}'(B)$ as follows:

Let $\Delta(B) = \prod\{\mathcal{G}(p) : p \in D \cap B\}$ and for each $\delta = \langle G(p) \rangle \in \Delta(B)$, let $G(\delta) = \bigcup\{G(p) : p \in D \cap B\}$. Define $\mathcal{W}'(B) = \{B \cup G(\delta) : \delta \in \Delta(B)\}$. Then from (1) and (2) it is easily seen that $\{\mathcal{W}'(B) : B \in \mathcal{B}\}$ has the following properties:

(3) $\mathcal{W}' = \bigcup\{\mathcal{W}'(B) : B \in \mathcal{B}\}$ is a CP family of closed subsets of X ;

(4) for each $W \in \mathcal{W}'(B)$, $(W \setminus M)^- = W$;

(5) if $B \subset U \in \tau(X)$ and $B \in \mathcal{B}$, then

there exists $W \in \mathcal{W}'(B)$ such that $B \subset W \subset U$.

By Lemma 1.2.7, we can take the L2-extension $\{\mathcal{W}''(W') : W' \in \mathcal{W}'\}$ of \mathcal{W}' in X .

For each $B \in \mathcal{B}$, we define $\mathcal{W}(B)$ as follows:

$$\mathcal{W}(B) = \bigcup \{\mathcal{W}''(W') : W' \in \mathcal{W}'(B)\}.$$

Then it is easy from (3), (4) and (5) to see that $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ has the required properties. □

We call $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ the *L3-extension of \mathcal{B} to X in Y* .

Lemma 1.2.10. *Let M be a closed subset of an M_3 -space X with property (P) such that $X \setminus M$ is dense in X . Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of closed subsets of X satisfying the following:*

(i) $\mathcal{W} = \bigcup \{\mathcal{W}(B) : B \in \mathcal{B}\}$ is CP in X ;

(ii) for each $W \in \mathcal{W}$,

$$(\text{Int } W)^- \cap M = W \cap M;$$

(iii) if $B \subset U \in \tau(X)$ and $B \in \mathcal{B}$, then $B \subset W \subset U$ for some $W \in \mathcal{W}(B)$.

Proof. Let $\{\mathcal{W}'(B \cap M) : B \in \mathcal{B}\}$ be the L3-extension of $\mathcal{B}|M$ to X in X . For each $B \in \mathcal{B}$, let

$$\mathcal{W}(B) = \{B \cup W : W \in \mathcal{W}'(B \cap M)\}.$$

Then it is easy to see that $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ has the required properties. □

We call $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ the *L4-extension of \mathcal{B} in X with respect to M* .

Lemma 1.2.11. *Let M and X_1 be closed subsets of an M_3 -space X with property (P) such that $X_1 \subset M$ and $M \setminus X_1 \subset (X \setminus M)^-$. Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of closed subsets of X satisfying the following:*

(i) $\bigcup\{\mathcal{W}(B) : B \in \mathcal{B}\}$ is CP in X ;

(ii) for each $B \in \mathcal{B}$ and each $W \in \mathcal{W}(B)$, $W \cap X_1 = B \cap X_1$ and

$$(\text{Int } W)^- \cap (M \setminus X_1) = W \cap (M \setminus X_1);$$

(iii) if $B \setminus X_1 \subset O \in \tau(X)$, $B \in \mathcal{B}$, then there exists $W \in \mathcal{W}(B)$ such that $B \subset W$ and $W \setminus X_1 \subset O$.

Proof. First we observe that property (P) is hereditary with respect to open subspaces and we recall that M_3 -spaces are hereditary. Thus we can apply Lemma 1.2.10 to the open subspace $X \setminus X_1$. There exists the L4-extension $\{\mathcal{W}'(B \cap (X \setminus X_1)) : B \in \mathcal{B}\}$ of $\mathcal{B}|(X \setminus X_1)$ in $X \setminus X_1$ with respect to $M \setminus X_1$. For each $B \in \mathcal{B}$, we define $\mathcal{W}(B)$ as follows:

$$\mathcal{W}(B) = \{B \cup W : W \in \mathcal{W}'(B \cap (X \setminus X_1)) \text{ and } W \cap T(X \setminus B)^- = \emptyset\},$$

where T is the L1-operator with respect to X_1 in X . Then it is easy to see that $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ satisfies all conditions (i), (ii) and (iii). \square

We call $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ the L5-extension of \mathcal{B} in X with respect to $\{M, X_1\}$.

Lemma 1.2.12. *Let $\{X(n) : n \in N\}$ be a disjoint cover of an M_3 -space X with property (P) satisfying the following:*

For each n ,

(a) $\bigcup\{X(t) : t \leq n\}$ is closed in X ;

(b) $X(n) \subset (\bigcup\{X(t) : t > n\})^-$.

(From here, for brevity we call such $\{X(n)\}$ *the special partition of X*.)

Let \mathcal{B} be a CP family of closed subsets of X . Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of X satisfying the following:

- (i) $\bigcup\{\mathcal{W}(B) : B \in \mathcal{B}\}$ is a CP family of regular closed subsets of X ;
- (ii) if $B \subset U \in \tau(X)$ and $B \in \mathcal{B}$, then there exists $W \in \mathcal{W}(B)$ such that $B \subset \text{Int } W \subset W \subset U$.

Proof. By Lemma 1.2.7 with $M = \emptyset$, there exists a collection $\{\mathcal{W}(B; 1) : B \in \mathcal{B}\}$ of families of X satisfying the following:

- (1) $\mathcal{W}(1) = \bigcup\{\mathcal{W}(B; 1) : B \in \mathcal{B}\}$ is a CP family of closed subsets of X and
for each $B \in \mathcal{B}$, $\mathcal{W}(B; 1)$ is a closed neighborhood base of B in X .

By Lemma 1.2.10, there exists the L4-extension $\{\mathcal{W}(W_1; 2) : W_1 \in \mathcal{W}(1)\}$ of $\mathcal{W}(1)$ in X with respect to $X(1)$, which satisfies the following:

- (2) $\mathcal{W}(2) = \bigcup\{\mathcal{W}(W_1; 2) : W_1 \in \mathcal{W}(1)\}$ is a CP family of closed subsets of X ;
- (3) for each $W_2 \in \mathcal{W}(2)$,
 $(\text{Int } W_2)^- \cap X(1) = W_2 \cap X(1)$;

- (4) if $W_1 \subset U \in \tau(X)$ and $W_1 \in \mathcal{W}(1)$, then

there exists $W_2 \in \mathcal{W}(W_1; 2)$ such that $W_1 \subset W_2 \subset U$.

Let T_1 be the L1-operator with respect to $X(1)$ in X and construct the family

$$\mathcal{T}(1) = \{T_1(X \setminus M) : W \in \mathcal{W}(2)\}.$$

Assume that we have constructed families

$$\begin{aligned} \mathcal{W}(n) &= \bigcup\{\mathcal{W}(W; n) : W \in \mathcal{W}(n-1)\}, \\ \mathcal{T}(n-1) &= \{T_{n-1}(X \setminus W) : W \in \mathcal{W}(n)\}, \end{aligned}$$

where T_{n-1} is the L1-operator with respect to $\bigcup\{X(t) : t \leq n-1\}$ in X . We apply Lemma1.2.11 to the case

$$M = \bigcup\{X(t) : t \leq n\}, \quad X_1 = \bigcup\{X(t) : t \leq n-1\}$$

and $\mathcal{B} = \mathcal{W}(n)$ in X . Thus there exists the L5-extension $\{\mathcal{W}(W; n+1) : W \in \mathcal{W}(n)\}$ of $\mathcal{W}(n)$ in X with respect to $\{M, X_1\}$ replaced with the above, which satisfies the following:

$$(5) \quad \mathcal{W}(n+1) = \bigcup\{\mathcal{W}(W; n+1) : W \in \mathcal{W}(n)\}$$

is a CP family of closed subsets of X ;

$$(6) \quad \text{for each } W_2 \in \mathcal{W}(W_1; n+1) \text{ and } W_1 \in \mathcal{W}(n),$$

$$(\text{Int } W_2)^- \cap X(n) = X_2 \cap X(n) \text{ and}$$

$$W_2 \cap \left(\bigcup\{X(t) : t \leq n-1\} \right) = W_1 \cap \left(\bigcup\{X(t) : t \leq n-1\} \right);$$

$$(7) \quad \text{if } W_1 \setminus X_1 \subset O \in \tau(X) \text{ and } W_1 \in \mathcal{W}(n), \text{ then}$$

there exists $W_2 \in \mathcal{W}(W_1; n+1)$ such that $W_1 \subset W_2$ and $W_2 \setminus X_1 \subset O$.

By Lemma1.2.6, we take the family

$$\mathcal{T}(n) = \{T_n(X \setminus W) : W \in \mathcal{W}(n+1)\},$$

where T_n is the L1-operator with respect to $\bigcup\{X(t) : t \leq n\}$ in X . With these preliminaries, we construct the final families $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ as follows: Let $B \in \mathcal{B}$ be fixed. Let $\Delta(B)$ be the totality of

$$\delta = \langle W(n) \rangle \in \prod\{\mathcal{W}(n) : n \in N\}$$

satisfying the following two conditions:

$$(8) \quad W(1) \in \mathcal{W}(B; 1) \text{ and } W(n+1) \in \mathcal{W}(W(n); n+1) \text{ for each } n;$$

$$(9) \quad W(n+2) \cap \left(\bigcup \{T_i(X \setminus W(i+1))^- : i \leq n\} \right) \cap \left(\bigcup \{X(t) : t \geq n+1\} \right) = \emptyset,$$

for each n .

For each $\delta = \langle W(n) \rangle \in \Delta(B)$, let $W(\delta) = \bigcup \{W(n) : n \in N\}$ and let $\mathcal{W}(B) = \{W(\delta) : \delta \in \Delta(B)\}$. Then we show that $\mathcal{W}(B)$ has the following property:

$$(10) \quad \text{Each } W(\delta) \in \mathcal{W}(B) \text{ is regular closed in } X \text{ and a neighborhood of } B \text{ in } X.$$

To see that $W(\delta)$ is closed in X , let $p \in (X \setminus W(\delta)) \cap X(n)$; then $p \in T_n(X \setminus W(n+1))$ which is an open subset of X missing $W(\delta)$ from (9). On the other hand, by (3), (6) and the definition of $W(\delta)$ we easily have

$$W(\delta) = \left(\bigcup \{\text{Int } W(n) : n \in N\} \right)^-.$$

Since $W(1) \subset W(\delta)$ and by (1) $W(1)$ is a neighborhood of B in X , $W(\delta)$ is also a neighborhood of B in X . This shows (10).

By the similar way to the above, the following is easy to see:

$$(11) \quad \bigcup \{\mathcal{W}(B) : B \in \mathcal{B}\} \text{ is CP in } X.$$

By (1), (2) and (7), it is easy to see that $\mathcal{W}(B)$ is a neighborhood base of B in X . This completes the proof. \square

Remark 3. Taking it into account that property (P) is hereditary with respect to open subspaces, we can say the following which is needed in the proof of the next theorem: Let X , $\{X(n)\}$ and \mathcal{B} be the same as above except for that X is an open subspace of an M_3 -space Y with property (P). Then there exists a collection $\{\mathcal{W}(B) : B \in \mathcal{B}\}$ of families of X satisfying the same (i) and (ii) as above.

Lemma 1.2.13. *Let Z be an open subspace of an M_3 -space X with property (P). Let \mathcal{B} be a CP family of closed subsets of the subspace Z^- and $\{O(B) : B \in \mathcal{B}\}$ a family of subsets of Z^- such that $O(B)$ is open in Z^- and $B \subset O(B)$ for each $B \in \mathcal{B}$. Let \mathcal{V} be a disjoint open cover of Z . Then there exists a family $\{W(B) : B \in \mathcal{B}\}$ of Z^- satisfying the following:*

(i) $\{W(B) : B \in \mathcal{B}\}$ is a CP family of regular closed subsets of Z^- ;

(ii) for each $B \in \mathcal{B}$, $W(B) \cap \partial Z \subset O(B) \cap \partial Z$ and

$$B \subset W(B) \subset \text{St}(O(B), \mathcal{V}).$$

Proof. By Lemma 1.2.9, there exists a the L3-extension $\{\mathcal{W}(B \cap \partial Z) : B \in \mathcal{B}\}$ of $\mathcal{B}|\partial Z$ to Z^- in X , which satisfies the following:

(1) $\mathcal{W} = \bigcup \{\mathcal{W}(B \cap \partial Z) : B \in \mathcal{B}\}$ is a CP family of closed subsets of Z^- and

for each $W \in \mathcal{W}$, $(\text{Int } W)^- \cap \partial Z = W \cap \partial Z$;

(2) if $B \subset U \in \tau(X)$, $B \in \mathcal{B}$, then $W \subset U$ for some $W \in \mathcal{W}(B \cap \partial Z)$.

For each $B \in \mathcal{B}$, by (2) we take $W_1(B) \in \mathcal{W}(B \cap \partial Z)$ such that $W_1(B) \subset O(B)$. By (1) and CP-ness of \mathcal{B} , we see that

$$\{G(B) = B \cup W_1(B) : B \in \mathcal{B}\}$$

is a CP family of closed subsets of Z^- . Thus, by Fact 1.2.5 there exists a σ -closed-discrete subset D of Z^- such that for each $B \in \mathcal{B}$

$$G(B) = (G(B) \cap D)^-.$$

Let $D = \bigcup \{D_n : n \in N\}$, where each D_n is a discrete and closed subset of Z^- . For each $p \in D_n \cap Z$, $n \in N$, we take an open neighborhood $O(p)$ of p in X such that

(3) $p \in O(p) \subset O(p)^- \subset S(\{p\}, n) \cap V(p)$

where $V(p)$ is the unique member of \mathcal{V} with $p \in V(p)$. Moreover, we can assume that

(4) $\{O(p) : p \in D_n \cap Z\}$ is discrete in Z^- .

For each $B \in \mathcal{B}$, we define $W(B)$ as follows:

$$W(B) = G(B) \cup \left(\bigcup \{O(p)^- : p \in D \cap G(B) \cap Z\} \right).$$

Then it is easy to see that $\{W(B) : B \in \mathcal{B}\}$ has the required properties. \square

1.2.3 The main result.

Theorem 1.2.14. *If X is an M_3 -space with property (P), then X is an M_1 -space such that every closed subset of X has a CP open neighborhood base in X .*

Proof. Let X be an M_3 -space with property (P). Then it suffices to show that every closed subset of X has a CP neighborhood base whose members are regular closed in X , [BL74, Remark 2.7]. Let M be a closed subset of X . By Fact 1.2.1, there exists a CP closed neighborhood base \mathcal{B} of M in X . By Fact 1.2.4, there exists the tile \mathcal{F} on \mathcal{B} such that $\mathcal{F} = \bigcup\{\mathcal{F}(n) : n \in N\}$, where each $\mathcal{F}(n)$ is a discrete family of closed subsets of X . For each n , let

$$\begin{aligned} X'(1) &= \bigcup \mathcal{F}(1), \\ X'(n+1) &= \bigcup \mathcal{F}(n+1) \setminus (X'(1) \cup \dots \cup X'(n)), \\ Z(n) &= \text{Int } X'(n) \end{aligned}$$

and let

$$Z = \bigcup\{Z(n) : n \in N\}, \quad Y = X \setminus Z^- \quad \text{and} \quad X(n) = X'(n) \cap Y.$$

Then $\{X(n) : n \in N\}$ is the special partition of the open subspace Y (see Lemma 1.2.12). We apply the Remark 3 to a CP family $\mathcal{B}|Y$ of closed subsets of Y . Then there exists a collection $\{\mathcal{W}(B \cap Y) : B \in \mathcal{B}\}$ of families of Y satisfying the following:

(1) $\bigcup\{\mathcal{W}(B \cap Y) : B \in \mathcal{B}\}$ is a CP family of regular closed subsets of Y ;

(2) for each $B \in \mathcal{B}$, if $B \cap Y \subset U \in \tau(X)$, then

$$\text{there exists } W \in \mathcal{W}(B \cap Y) \text{ such that } B \cap Y \subset \text{Int } W \subset W \subset U.$$

Moreover without loss of generality we can assume the following:

(3) For each $B \in \mathcal{B}$, $\left(\bigcup \mathcal{W}(B \cap Y)\right) \cap T(X \setminus B)^- = \emptyset$,

where T is the L1-operator with respect to Z^- in X . Let Δ be the totality of pairs $\delta = (B_1, B_2)$ of members of \mathcal{B} such that $B_1 \subset \text{Int } B_2$. By Lemma 1.2.13, for each $\delta = (B_1, B_2) \in \Delta$, there exists a subset $W(\delta)$ of the subspace Z^- satisfying the following:

$$(4) \quad \{W(\delta) : \delta \in \Delta\} \text{ is CP in } Z^-;$$

(5) for each $\delta = (B_1, B_2) \in \Delta$, $W(\delta)$ is a regular closed subset of Z^- such that

$$B_1 \cap Z^- \subset W(\delta) \subset \text{St}(\text{Int } B_2, \mathcal{V}),$$

where $\mathcal{V} = \mathcal{F}|Z$ is a disjoint open cover of Z . Note that the last inclusion relation implies $W(\delta) \subset B_2$. For each pair $\delta = (B_1, B_2) \in \Delta$, define

$$\mathcal{W}(\delta) = \{W \cup W(\delta) : W \in \mathcal{W}(B_1 \cap Y)\}.$$

By the statements (1) to (5), it is easy to see that $\{\mathcal{W}(\delta) : \delta \in \Delta\}$ has the following properties:

$$(6) \quad \mathcal{W} = \bigcup \{\mathcal{W}(\delta) : \delta \in \Delta\}$$

is a CP family of regular closed neighborhoods of M in X ;

(7) if $M \subset O \in \tau(X)$, then there exists $\delta \in \Delta$ and $W \in \mathcal{W}(\delta)$ such that

$$M \subset \text{Int } W \subset W \subset O.$$

Hence we have obtained a CP neighborhood base of M consisting of regular closed subsets of X . This completes the proof. \square

Theorem 1.2.15. *A k - M_3 -space has property (P), and therefore a k - M_3 -space is M_1 .*

Proof. Note that for an M_3 -space both k -ness and sequentialness are equivalent, because every compact subset of an M_3 -space is metrizable [Gru84, Theorem 2.13].

So, it suffices to show that a sequential M_3 -space has property (P). Since every closed subspace of a sequential M_3 -space is also sequential M_3 , it suffices to show the following: For a closed nowhere dense subset M of a sequential M_3 -space X and any point $p \in M$, there exists a CP family $\mathcal{G}(p)$ of closed subsets of X that acts as a local net at the point p in X and that has $(G \setminus M)^- = G$ for each $G \in \mathcal{G}(p)$.

For contradiction, suppose that there exists a point $p \in M$ for which there is no CP family $\mathcal{G}(p)$ of the type as described above. We show that M cannot be nowhere dense in X , and that contradiction completes the proof.

We say that point $x \in X$ “has $\mathcal{G}(x)$ ” provided $\mathcal{G}(x)$ is a CP family of closed subsets of X such that $\mathcal{G}(x)$ acts as a local net at x and for each $G \in \mathcal{G}(x)$ we have $(G \setminus M)^- = G$. If no such family exists, we say that x does not have $\mathcal{G}(x)$.

Claim 1: Suppose that $x \in X$ does not have $\mathcal{G}(x)$ and that $\{z(m) : m \in N\}$ is any sequence of points of X that converges to x . Then for some n , $m \geq n$ implies that $z(m)$ does not have $\mathcal{G}(z(m))$.

Proof of the claim: Suppose that the claim is false. Then for some $\{z(n) : n \in N\}$ converging to x , there is an infinite subset $N' \subset N$ such that for each $n \in N'$, $z(n)$ has $\mathcal{G}(z(n))$. Using the given sequence $\{z(n)\}$ that converges to x , by replacing N' by an infinite subset N' if necessary, we may assume that for each $n \in N'$, we have

$$\{z(m) : m \geq n, m \in N\} \subset S(\{x\}, n)$$

and

$$\bigcup \mathcal{G}(z(n)) \subset S(\{x\}, n).$$

Set

$$\Delta(n) = \prod \{\mathcal{G}(z(k)) : k \geq n, k \in N'\}, n \in N',$$

and for each $\delta = \langle G(z(k)) \rangle \in \Delta(n)$, set

$$G(\delta) = \left(\bigcup \{G(z(k)) : k \geq n\} \right)^-.$$

Letting

$$\mathcal{G}(n) = \{G(\delta) : \delta \in \Delta(n)\}, \quad n \in N',$$

we define the family $\mathcal{G}(x)$ as

$$\mathcal{G}(x) = \bigcup \{\mathcal{G}(n) : n \in N'\}.$$

Then, as easily seen, $\mathcal{G}(x)$ is a family of closed subsets of X that acts as a local net at x in X and having $(G \setminus M)^- = G$ for each $G \in \mathcal{G}(x)$. We show that $\mathcal{G}(x)$ is CP in X . Let $\mathcal{G}_0 \subset \mathcal{G}(x)$ and suppose

$$y \in X \setminus \left(\bigcup \mathcal{G}_0 \right).$$

Since $x \neq y$, there exists $k \in N'$ such that $y \notin S(\{x\}, k)^-$. Let $\mathcal{G}_0 = \bigcup \{\mathcal{G}_0(n) : n \in N'\}$, where $\mathcal{G}_0(n) \subset \mathcal{G}(n)$ for each $n \in N'$. If we note that

$$\bigcup \{\mathcal{G}(z(t)) : t \geq k, t \in N'\} \subset S(\{x\}, k)$$

and that

$$\bigcup \{\mathcal{G}(z(t)) : t < k, t \in N'\}$$

is CP in X , then it is easy to find a neighborhood O of y in X such that

$$O \cap \left(\bigcup \mathcal{G}_0 \right) = \emptyset,$$

which proves $y \notin \left(\bigcup \mathcal{G}_0 \right)^-$. Hence $\mathcal{G}(x)$ is CP in X . But this contradicts the key property of x as described in Claim 1. Thus Claim 1 holds.

Now, let $T_0 = \{p\}$. Apply Claim 2 to the point p to show that for each sequence $\{z(m) : m \in N\}$ of points of X converging to p there is $n \in N$ beyond which no term $z(m)$ of the sequence has $\mathcal{G}(z(m))$. Let $\{Z_\alpha : \alpha \in A(1)\}$ be the totality of sequences $Z_\alpha = \{z_\alpha(n) : n \in N\}$ in X that converges to p and have the property that no term $z_\alpha(n)$ has $\mathcal{G}(z_\alpha(n))$. Let

$$T_1 = \bigcup \{Z_\alpha : \alpha \in A(1)\} \cup T_0.$$

Suppose $n \geq 1$ and we have defined the set T_n . Let $\{Z_\alpha : \alpha \in A(n+1)\}$ be the totality of sequences $Z_\alpha = \{z_\alpha(n) : n \in N\}$ in X that converges to a point of T_n and have the property that no term $z_\alpha(n)$ of Z_α has $\mathcal{G}(z_\alpha(n))$. Define

$$T_{n+1} = \bigcup \{Z_\alpha : \alpha \in A(n+1)\} \cup T_n.$$

With T_n defined recursively in this manner, let $T = \bigcup \{T_n : n \geq 0\}$.

Claim 2: T is sequentially open in X .

For suppose that $\{y(n) : n \in N\}$ is a sequence of points of X that converges to some point $q \in T$. Then $q \in T_j$ for some j . According to Claim 1, there is $m \in N$ such that for each $n \geq m$, the point $y(n)$ does not have $\mathcal{G}(y(n))$. Hence the sequence $Z = \{y(m+k) : k \in N\}$ is one of the sequences Z_α for some $\alpha \in A(j+1)$ so that $Z \subset T(j+1) \subset T$. Hence T is sequentially open in X .

Because X is sequential, we conclude that T is an open subset of X .

Claim 3: $T \subset M$.

This claim follows easily from the observation that if $y \in X \setminus M$, y has a CP closed local net \mathcal{G} consisting of sets that entirely miss the closed set M . Then for each $G \in \mathcal{G}$, we certainly have $(G \setminus M)^- = G$.

We have a contradiction to the original observation that M is closed and nowhere dense in Y . This completes the proof. □

1.3 A Lašnev space is LF-netted

We show that every Lašnev space is LF-netted.

1.3.1 Introduction.

All spaces are assumed to be regular T_2 -spaces. For a space X , $\tau(X)$ denotes the topology of X . \mathbb{N} always denotes natural numbers. For a subset A of a space X , let A^- and $IntA$ be the closure and interior of A , respectively, in X . For simplicity, let letters CP, LF stand for closure-preserving, locally finite, respectively. For a family \mathcal{U} of subsets of X and a subset A of X , $\mathcal{U}|A$ denotes the family $\{U \cap A | U \in \mathcal{U}\}$ and \mathcal{U}^- denotes the family $\{U^- | U \in \mathcal{U}\}$. We call \mathcal{U} a *neighborhood base of A in X* if \mathcal{U} consists of neighborhoods of A in X and if $A \subset O \in \tau(X)$, there exists $U \in \mathcal{U}$ such that $A \subset U \subset O$. In addition, if all member of \mathcal{U} are closed, open in X , then \mathcal{U} is called *closed*, *open*, respectively.

Recently, Junnila and Yajima [JY] introduced a new class of LF-netted spaces. A space X is defined to be *LF-netted* if X has a net \mathcal{F} that is both σ -LF and LF-regular in X , where \mathcal{F} is called *LF-regular* in X if for each closed subset S of X , $\{F \in \mathcal{F} | F \cap S \neq \emptyset\}$ is LF in $X \setminus S$. They introduced it in search of theory of normality of product spaces rather than generalized metric spaces. However, as for the results on generalized metric spaces, they gave some relations between LF-netted spaces, σ -spaces and M_3 -, \mathcal{F}_σ -metrizable spaces. We study this class from the point of view of generalized metric spaces, especially here we establish the implication:

$$\text{Lašnev space} \longrightarrow \text{LF-netted space.}$$

This is the positive answer to [JY, Problem 4*]. Regarding to this problem, Sakai, Tamano and Yajima proved recently that every separable Lašnev space is LF-netted [STY, Theorem 3.8]. But our proof is different from theirs in the sense that actually every D-space is shown to be LF-netted. Here, M_3 -spaces are used to mean spaces having both the stratification and a σ -CP quasi-base.

1.3.2 Lašnev spaces and LF-netted spaces.

We state known definitions and known facts used here:

Definition 1.3.1 ([Nag80b, Definition 4.4]). A space X is called a *D-space* if X is a paracompact σ -space such that each closed subset F of X has a uniformly approaching anti-cover in X , where an open cover \mathcal{U} of $X \setminus F$ is called *uniformly approaching* to F in X if for each $O \in \tau(X)$,

$$\text{St}(X \setminus O, \mathcal{U})^- \cap (O \cap F) = \emptyset.$$

Fact 1.3.2. Let X be a space and F a closed subset of X which has a uniformly approaching anti-cover \mathcal{W} of F . Then for each open subset U of X , there exists $\mathcal{W}_0 \subset \mathcal{W}$ such that $(F \cap U) \cup (\bigcup \mathcal{W}_0)$ is open in X and

$$\left[(F \cap U) \cup \left(\bigcup \mathcal{W}_0 \right) \right] \cap \text{St}(X \setminus U, \mathcal{W}) = \emptyset.$$

Fact 1.3.3. A space X is a stratifiable space (or equivalently an M_3 -space) if and only if there exists a stratification $S : \{\text{closed subsets of } X\} \times \mathbb{N} \rightarrow \tau(X)$ satisfying the following:

(i) For each closed subset F of X ,

$$F = \bigcap \{S(F, n) \mid n \in \mathbb{N}\} = \bigcap \{S(F, n)^- \mid n \in \mathbb{N}\};$$

(ii) if F, F' are closed in X and $F \subset F'$, then $S(F, n) \subset S(F', n)$ for each n ;

(iii) for each n and each F ,

$$S(F, n+1)^- \subset S(F, n).$$

(Originally, the term stratification is used for S with (i) and (ii), but one with (i), (ii) and (iii) follows by a slight change.)

Fact 1.3.4 (Essentially in [SN68]). Let \mathcal{B} be a CP family of closed subsets of an M_3 -space X . Then there exists a pair $\langle \mathcal{F}, \mathcal{V} \rangle$ of families of subsets of X satisfying the following:

- (i) \mathcal{F} is a star-finite, σ -discrete closed cover of X such that if $F \in \mathcal{F}$ and $B \in \mathcal{B}$, then $B \cap F \neq \emptyset$ if and only if $F \subset B$;
- (ii) $\mathcal{V} = \{V(F) | F \in \mathcal{F}\}$ is a point-finite, σ -discrete open cover of X such that $F \subset V(F)$ for each $F \in \mathcal{F}$ and such that if $F \cap B = \emptyset$, $F \in \mathcal{F}$, $B \in \mathcal{B}$, then $V(F) \cap B = \emptyset$.

Since our proof of Theorem 1.3.6 depends on the construction of $\langle \mathcal{F}, \mathcal{V} \rangle$ rather than the fact itself, we sketch it roughly: Let $\mathcal{P} = \{P(\delta) | \delta \in \Delta\}$ be the partition of X by $\mathcal{B}' = \mathcal{B} \cup \{X\}$, that is, for each $\delta \in \Delta$,

$$P(\delta) = \bigcap \mathcal{B}(\delta) \setminus \bigcup (\mathcal{B}' \setminus \mathcal{B}(\delta)),$$

where $\mathcal{B}(\delta) \subset \mathcal{B}'$. For each $n \in \mathbb{N}$, let

$$F(\delta, n) = P(\delta) \cap (S(T(\delta), n-1)^- \setminus S(T(\delta), n)),$$

where $T(\delta) = \bigcup (\mathcal{B}' \setminus \mathcal{B}(\delta))$ and $S(\cdot, 0) = X$. Then it is easy to check that $\mathcal{F}(n) = \{F(\delta, n) | \delta \in \Delta\}$ is a discrete family of closed subsets of X and $\mathcal{F} = \bigcup \{\mathcal{F}(n) | n \in \mathbb{N}\}$ is star-finite. Since X is paracompact, there exists a discrete open expansion

$$\mathcal{V}(n) = \{V(\delta, n) | \delta \in \Delta\}, \quad n \in \mathbb{N},$$

of $\mathcal{F}(n)$ such that

$$F(\delta, n) \subset V(\delta, n) \subset S(F(\delta, n), n) \cap (S(T(\delta), n-2) \setminus S(T(\delta), n+1)^-)$$

for each $\delta \in \Delta$, $n \in \mathbb{N}$, where $S(\cdot, -1) = X$. Then it is easy to see that $\mathcal{V} = \bigcup \{\mathcal{V}(n) | n \in \mathbb{N}\}$ is the required one.

Fact 1.3.5. Let \mathcal{W} be a LF family of open subsets of a space X . Then there exists a LF closed cover \mathcal{F} of X such that if $F \cap W \neq \emptyset$, $F \in \mathcal{F}$, $W \in \mathcal{W}$, then $F \subset W^-$.

Proof. Let $\{P(\delta) | \delta \in \Delta\}$ be the partition of X by $\mathcal{W}^- \cup \{X\}$. Then it is easy to see that $\mathcal{F} = \{P(\delta)^- | \delta \in \Delta\}$ is a required family. \square

Theorem 1.3.6. *Every D-space is an M_1 -, LF-netted space.*

Proof. Let X be a D-space. Since it is an M_1 -space [Nag80b], it suffices to show that X is LF-netted. Since X is an M_3 -space, for each $U \in \tau(X)$ there exists a sequence $\{B_n(U) | n \in \mathbb{N}\}$ of subsets of X satisfying the following:

(1) For each n ,

$$\mathcal{B}(n) = \{B_m(U) | U \in \tau(X), m \leq n\}$$

is a CP family of closed subsets of X ;

(2) for each $U \in \tau(X)$,

$$U = \bigcup \{B_n(U) | n \in \mathbb{N}\} = \bigcup \{\text{Int } B_n(U) | n \in \mathbb{N}\}.$$

Let $n \in \mathbb{N}$ be fixed for a while. Because of (1), we can use Fact 1.3.4 to $\mathcal{B}(n)$ and get a pair $\langle \mathcal{F}(n), \mathcal{V}(n) \rangle$ of families of subsets of X satisfying the following:

(3) $\mathcal{F}(n) = \bigcup \{\mathcal{F}(n, m) | m \in \mathbb{N}\}$

is a star-finite closed cover of X such that each $\mathcal{F}(n, m)$ is discrete in X and if $B \in \mathcal{B}(n)$, $F \in \mathcal{F}(n)$, then $F \cap B \neq \emptyset$ if and only if $F \subset B$;

(4) $\mathcal{V}(n) = \{V(F) | F \in \mathcal{F}(n)\}$ is a point-finite open cover of X such that

if $B \cap F = \emptyset$, $B \in \mathcal{B}(n)$, $F \in \mathcal{F}(n)$, then $B \cap V(F) = \emptyset$ and such that

for each m , $\{V(F) | F \in \mathcal{F}(n, m)\}$ is a discrete open expansion of $\mathcal{F}(n, m)$.

Since X is a D-space, each $F \in \mathcal{F}(n)$ has a uniformly approaching anti-cover $\mathcal{W}'(F)$ in X . Since X is hereditarily paracompact and F is a G_δ -set of X , we can assume the following:

(5) $\mathcal{W}'(F)$ is LF in $X \setminus F$ and σ -LF in X .

Let $\mathcal{W}(F) = \mathcal{W}'(F) | V(F)$. We inductively construct $\{\mathcal{H}(n, m) | m \in \mathbb{N}\}$ as follows:

For $m = 1$, let $\mathcal{H}(n, 1) = \mathcal{F}(n, 1)$. We can easily observe by (5) that for each $m > 1$ and $F \in \mathcal{F}(n, m)$ the family

$$\mathcal{W}^*(F) = \bigcup \left\{ \mathcal{W}(F', F) \mid F' \in \bigcup \{ \mathcal{F}(n, i) \mid i < m \}, F \cap F' = \emptyset \right\},$$

where $\mathcal{W}(F', F) = \mathcal{W}(F')|_F$, is a LF family of open subsets of the subspace F . By Fact 1.3.5, there exists a LF closed cover $\mathcal{H}_n(F)$ of F satisfying the following:

$$(6) \quad \text{If } W \in \mathcal{W}^*(F), H \in \mathcal{H}_n(F), \text{ then } W \cap H \neq \emptyset \text{ implies } H \subset W^-.$$

Let

$$\begin{aligned} \mathcal{H}(n, m) &= \bigcup \{ \mathcal{H}_n(F) \mid F \in \mathcal{F}(n, m) \}, \\ \mathcal{H}(n) &= \bigcup \{ \mathcal{H}(n, m) \mid m \in \mathbb{N} \}. \end{aligned}$$

Then $\mathcal{H}(n)$ is a σ -LF closed cover of X . By (2), $\mathcal{H} = \bigcup \{ \mathcal{H}(n) \mid n \in \mathbb{N} \}$ is a net for X . We show that \mathcal{H} is LF-regular. Let $p \in U \in \tau(X)$. By (2), there exists $n \in \mathbb{N}$ such that $p \in \text{Int } B_n(U)$. By (3) and by the fact that $\mathcal{H}(m) < \mathcal{F}(m)$, $m \in \mathbb{N}$, it is easy to see that

$$B_n(U) \cap \text{St} \left(X \setminus U, \bigcup \{ \mathcal{H}(t) \mid t \geq n \} \right) = \emptyset.$$

Let $k < n$ be fixed. By the property of $\mathcal{F}(k)$ in (3), there exist $s_0, t_0 \in \mathbb{N}$ and $F \in \mathcal{F}(k, s_0)$ such that $p \in F$ and $F \cap F' = \emptyset$ for each $F' \in \bigcup \{ \mathcal{F}(k, t) \mid t \geq t_0 \}$. By Fact 1.3.2, there exists $\mathcal{W}_0 \subset \mathcal{W}(F)$ such that

$$W_1 = (U \cap F) \cup \left(\bigcup \mathcal{W}_0 \right)$$

is an open neighborhood of p in X such that

$$(7) \quad W_1 \cap \text{St}(X \setminus U, \mathcal{W}(F))^- = \emptyset.$$

This means $(\bigcup \mathcal{W}_0)^- \cap (X \setminus F) \subset U$. Let

$$\begin{aligned} W_2 &= X \setminus \bigcup \left\{ F' \in \bigcup \{ \mathcal{F}(k, t) \mid t < t_0 \} \mid p \notin F' \right\}, \\ \mathcal{F}(p) &= \left\{ F' \in \bigcup \{ \mathcal{F}(k, t) \mid t < t_0 \} \mid p \in F' \right\}. \end{aligned}$$

Then $\mathcal{F}(p)$ is finite. Since for each $F' \in \mathcal{F}(p)$, $\mathcal{H}_k(F')$ is LF in F' and hence in X , there exists an open neighborhood W_3 of p in X such that

$$\left\{ H \in \bigcup \{ \mathcal{H}_k(F') \mid F' \in \mathcal{F}(p) \} \mid H \cap W_3 = \emptyset \right\}$$

is finite. By virtue of (6) and (7), if we set

$$O_k(p) = W_1 \cap W_2 \cap W_3,$$

then $O_k(p)$ is an open neighborhood of p in X such that

$$\{ H \in \mathcal{H}(k) \mid H \cap (X \setminus U) \neq \emptyset, H \cap O_k(p) \neq \emptyset \}$$

is finite. Hence we find an open neighborhood

$$O = \bigcap \{ O_k(p) \mid k < n \} \cap \text{Int } B_n(U)$$

of p in X such that O intersects only finite members of \mathcal{H} intersecting $X \setminus U$. Hence \mathcal{H} is LF-regular in X . This completes the proof. \square

We note that every Lašnev space is a D-space. Because every metric space is obviously a D-space and by virtue of [Miz81, Lemma 2], the closed image of a D-space is also D-space. As the corollary, we have:

Corollary 1.3.7. *Every Lašnev space is LF-netted.*

1.4 The historical note

Ceder [Ced61] introduced M_1 , M_2 , and M_3 -spaces, the definitions of which are viewed as variations on the Nagata-Smirnov metrization theorem inspired by Michael's three notes on paracompactness [Mic53, Mic57, Mic59].

Definition 1.4.1. A space X is an M_1 -space if it has a σ -CP base.

Definition 1.4.2. A space X is an M_2 -space if it has a σ -CP quasi-base.

Definition 1.4.3. A space X is an M_3 -space if it has a σ -cushioned pair base.

It is easy to see $M_1 \rightarrow M_2 \rightarrow M_3$. Ceder proposed the problem whether the reverse relations hold or not in [Ced61]. Borges [Bor66] renamed an M_3 -space "stratifiable" in terms of "stratification" and studied the class.

Definition 1.4.4. A *stratification* of a space X is a mapping $S : \{\text{closed subsets of } X\} \times \mathbb{N} \rightarrow \tau(X)$ satisfying the following;

- (i) for every closed F , $\bigcap_n S(F, n) = \bigcap_n \overline{S(F, n)} = F$,
- (ii) if $F \subset H$, then $S(F, n) \subset S(H, n)$ for each n .

A space X is a *stratifiable space* if it has a stratification.

As the positive answer to the second of the reverse relations was given by Junnila [Jun78] and Gruenhagen [Gru76] independently. Thus the first remains unsolved and it became one of the most outstanding unsolved problems in general topology.

To this problem, the first partial answer was given by Slaughter Jr. [Sla73].

Theorem 1.4.5. *Every Lašnev space, that is, the closed image of a metric space, is an M_1 -space.*

His proof was dependent on the decomposition theorem and the inner characterization of Lašnev spaces due to Lašnev [Laš65], especially the Fréchetness of a

Lašnev space was essential. For now, it seems rather primitive and lengthy. For one reason, he had no concept of irreducible mappings. A mapping $f : X \rightarrow Y$ is called *irreducible* if there is no closed proper subset X' of X such that $f(X') = Y$. If we can use this concept, we can prove Theorem 1.4.5 without Fréchetness and more simply.

Nagami defined two classes of spaces, both of which are the generalization of Lašnev spaces. The former is that of L-spaces in his first paper [Nag80b] and the latter is of free L-spaces in the second paper [Nag80a]. Two classes played an important role in his dimension theory. The definitions are obtained by choosing good properties from Lašnev spaces:

Definition 1.4.6. A space X is an *L-space* if X is a paracompact σ -space such that every closed subset M of X has an approaching anti-cover \mathcal{U} in X . (An open cover of \mathcal{U} of $X \setminus M$ is called *approaching* if for each neighborhood U of M in X , $\overline{\text{St}(X \setminus U, \mathcal{U})} \cap M = \emptyset$.)

Since $X \setminus M$ is paracompact, without lose of generality we can assume that $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ is locally finite in $X \setminus M$. If we define a family \mathcal{V} as

$$\mathcal{V} = \left\{ V(\delta) = M \cup \left(\bigcup \{U_\alpha | \alpha \in \delta\} \right) \mid \delta \subset A \text{ and } V(\delta) \text{ is an open neighborhood of } M \text{ in } X \right\},$$

then \mathcal{V} is a CP open neighborhood base of M in X , which leads to the fact that an L-spaces has good properties with respect to closed images and heredity, however, they have one defect that they are not preserved even by two product. So, it is natural for Nagami to define the class of free L-spaces, because they are characterized as subspaces of the countable product of L-spaces. But, the original definition is as follows:

Definition 1.4.7. A space X is a *free L-space* if X is a paracompact space with a free L-structure $\langle \mathcal{F}, \{\mathcal{U}_F | F \in \mathcal{F}\} \rangle$ satisfying the following:

(i) \mathcal{F} is a σ -discrete family of closed subsets of X and each $F \in \mathcal{F}$ has an anti-cover \mathcal{U}_F .

(ii) If $p \in O \in \tau(X)$, then there exists $F_1, \dots, F_k \in \mathcal{F}$ and their canonical neighborhoods V_1, \dots, V_k , respectively, such that

$$p \in \bigcap_i F_i \subset \bigcap_i V_i \subset O.$$

A free L-space is an M_1 -space because it has a σ -CP base \mathcal{W} consisting of all finite intersections of sets of form

$$V = F \cup \left(\bigcup \{U_\alpha \mid \alpha \in \delta\} \right),$$

where $F \in \mathcal{F}$ and $\mathcal{U}_F = \{U_\alpha \mid \alpha \in A_F\}$ is a locally finite open cover of $X \setminus F$.

Gruenhage showed in [Gru78] that every M_3 -space which is the countable of closed and discrete subsets is an M_0 -space, that is, a space which has a σ -CP base consisting of clopen subsets. This situation was extended to F_σ -metrizable M_3 -spaces in his paper [Gru80], where a space X is called *F_σ -metrizable* if X is the countable union of closed metrizable subsets, and this is equivalent to Nagami's original σ -metric space [Nag71]. Gruenhage's method adopted here is nothing but the construction of a special g -function on $X \times \mathbb{N}$. As well known, an M_3 -space is characterized by g -functions as follows:

Fact 1.4.8. A space X is M_3 if and only if there exists a function $g : X \times \mathbb{N} \rightarrow \tau(X)$ satisfying the following;

(i) $g(x, n) \ni x$ for each $(x, n) \in X \times \mathbb{N}$,

(ii) $y \in g(x, n) \rightarrow g(y, n) \subset g(x, n)$,

(iii) if H is a closed subset of X and $x \notin H$, then there exists $n \in \mathbb{N}$ such that

$$x \notin \overline{\bigcup \{g(y, n) \mid y \in H\}}.$$

Generally speaking, the following is a standard way to construct a σ -CP closed quasi-base through g -function g as above:

$$\mathcal{G} = \left\{ X \setminus \bigcup \{g(x, n) \mid x \in X \setminus O\} \mid O \in \tau(X), n \in \mathbb{N} \right\}.$$

But

$$\left\{ X \setminus \bigcup \{g(x, n) \mid x \in X \setminus O\} \mid O \in \tau(X) \right\}$$

may fail to be the family of regular closed subsets of X , through it is CP in X . Gruenhagen defined the special g -function on a F_σ -metrizable M_3 -space so that \mathcal{G} has the this additional property.

It is known that the class of M_3 , F_σ -metrizable spaces is not countably productive [Nag71]. This leads to the following natural question: Is any subspace of the countable product of M_3 , F_σ -metrizable spaces M_1 ? Mizokami showed in terms of M -structures that this is true [Miz84]. According to Nagami [Nag70], a space is a μ -space if it is embedded into a countable product of paracompact F_σ -metrizable spaces. Later, Junnila and Mizokami showed that the class of M_3 -spaces with M -structures coincide with that of M_3 , μ -spaces and also with that of spaces embedded into a countable product of M_3 , F_σ -metrizable spaces [JM85]. The class of M_3 , μ -spaces is closed under subspaces and countable product. Moreover, this class have one good property that they are the perfect images of strongly zero-dimensional one in the class. Since an M_3 , μ -space has $\text{Ind} = 0$ if and only if it is an M_0 -space, it follows that every M_3 , μ -space is the perfect image of an M_0 -space. What kind of bases does any perfect image of an M_0 -space have as its inner characterization? Ohta gave a complete answer to this. According to his paper [Oht89], perfect images of an M_0 -spaces can be characterized as spaces having a σ -FCP fitting base. Of course, these bases are stronger than those of M_1 -spaces.

Lašnev spaces are included in the class of Fréchet M_3 -spaces. If we strengthen Fréchetness to first countability, we reach to the class of Nagata spaces.

Ito gave the positive answer to the open question whether every Nagata space is M_1 . Actually, in [Ito85] he showed that if each point of an M_3 -space X has a CP open local base, then every closed subset of X has a CP open neighborhood base in X , and hence X is M_1 . His technique is simple but has a few applications: Let M be a closed subset of an M_3 -space X with the property as above. Then M has a CP closed neighborhood base \mathcal{B} in X , for which we can take a dense subset $D = \bigcup_n D_n$ with each D_n closed and discrete in X such that for each $B \in \mathcal{B}$ $\overline{B \cap D} = B$. Since by the assumption each $p \in D_n$, $n \in \mathbb{N}$, has a CP local base $\mathcal{V}(p)$ such that $\bigcup \mathcal{V}(p) \subset S(\{p\}, n)$, we can expand each B to sets of the form

$$B \cup \left(\bigcup \{V(p) \mid V(p) \in \mathcal{V}(p) \text{ and } p \in D \cap B\} \right)$$

so that the totality of such sets forms a CP open neighborhood base of M in X , and by virtue of [BL74], X is M_1 .

Tamano used this technique in a slightly different way and established that every Baire, Fréchet, M_3 -space is M_1 . In fact, in [Tam89] he showed that Ito's argument can apply to show $M_3 \rightarrow M_1$, even if every point p of X has a CP family \mathcal{V} of open subsets of X such that $\{\overline{V} \mid V \in \mathcal{V}\}$ acts as a local network at p in X , which Baire Fréchet σ -space have.

The implication and the relation between all spaces treated here are given in Figure 1.1.

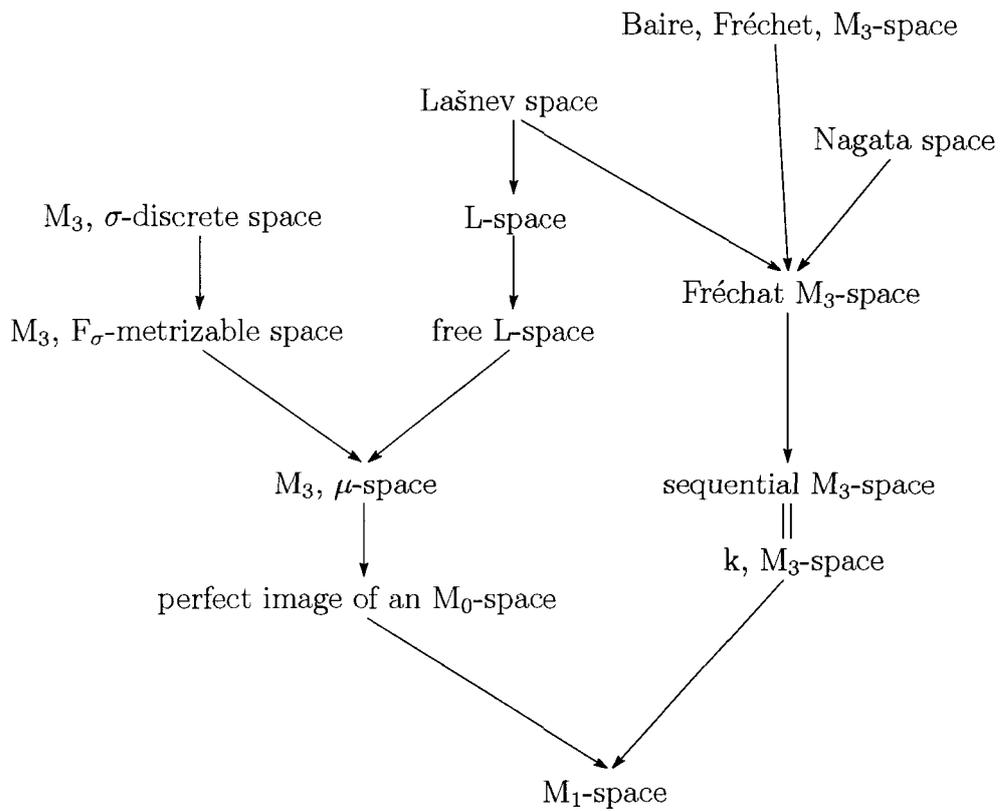


Figure 1.1: The implication and the relation

Chapter 2

On hyperspaces, mapping spaces and their relations to generalized metric spaces

2.1 Similarity of monotonical normality and stratifiability in hyperspaces and mapping spaces

We consider the coincidence of being monotonically normal and being stratifiable for hyperspaces $\mathcal{K}(X)$, $\mathcal{F}(X)$ and for mapping spaces $C_K(X, Y)$ with compact-open topology.

2.1.1 Introduction.

All spaces are assumed to be regular T_1 -spaces and all mappings to be continuous. The set of natural numbers is denoted by the letter \mathbb{N} .

A space X is called *monotonically normal* (for brevity in the sequel MN) if there exists a function G which assigns to each ordered pair (H, K) of disjoint closed subsets of X (or equivalently separated subsets of X [HLZ73, Lemma2.2]) an open subset $G(H, K)$ such that

- (a) $H \subset G(H, K) \subset G(H, K)^- \subset X \setminus K$;
- (b) if (H', K') is another pair having $H \subset H'$ and $K' \subset K$, then $G(H, K) \subset G(H', K')$.

(G is called a *monotone normality operator* (for brevity MN operator) for X .)

A space X is called *stratifiable* if there exists a function $S : \{\text{closed subsets of } X\} \times \mathbb{N} \rightarrow \tau_X$ (the topology of X), called a *stratification of X* , such that;

- (a) if H, K are closed subsets of X such that $H \subset K$, then $S(H, n) \subset S(K, n)$ for each $n \in \mathbb{N}$;
- (b) $H = \bigcap \{S(H, n) \mid n \in \mathbb{N}\} = \bigcap \{S(H, n)^- \mid n \in \mathbb{N}\}$ for each closed set H of X .

As known, a stratifiable space is equivalent to an M_2 -space i.e., a space having a σ -closure-preserving quasi-base.

In this section, we establish the similarity of being MN and being stratifiable for hyperspaces $\mathcal{K}(X)$, $\mathcal{F}(X)$ with finite topology and mapping spaces $C_K(X, Y)$ with compact-open topology. Until now, we have one form of similarity given by [HLZ73, Corollary4.2] that a space X is stratifiable if and only if X^{ω_0} , the product of countably many copy of X , is MN.

As for undefined term here refer to [Gru84].

2.1.2 Monotone normality of hyperspaces.

Let $\mathcal{K}(X)$ be the space consisting of all non-empty compact subsets of a space X with a finite topology which has the base

$$\{\langle U_1, \dots, U_k \rangle \mid U_1, \dots, U_k \in \tau_X, k \in \mathbb{N}\},$$

where $\langle A_1, \dots, A_k \rangle, A_1, \dots, A_k \subset X$, is the subset of $\mathcal{K}(X)$ defined by

$$\langle A_1, \dots, A_k \rangle = \left\{ K \in \mathcal{K}(X) \mid K \subset \bigcup_{i=1}^k A_i \text{ and } K \cap A_i \neq \emptyset \text{ for each } i \right\}.$$

If X is regular T_1 , then so is $\mathcal{K}(X)$ [Mic51, Theorem4.9.10]. Let $\mathcal{F}(X)$ be the subspace of $\mathcal{K}(X)$ consisting of all finite subsets of X .

To state the theorem, we introduce the following notation: If a space X has a convergent sequence $\{x_n \mid n \in \mathbb{N}\}$ of X with its limit point x such that $x_n \neq x$, $x_m \neq x_n$ for each $m, n \in \mathbb{N}$ with $m \neq n$, then we write

$$L(x) = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}.$$

Theorem 2.1.1. *Suppose that there exists $L(x), L(y)$ in a space X such that $L(x) \cap L(y) = \emptyset$. Then the followings are equivalent:*

(i) $\mathcal{F}(X)$ is MN.

(ii) X is stratifiable.

(iii) $\mathcal{F}(X)$ is stratifiable.

Proof. (ii) \rightarrow (iii) is shown in [Miz96, Theorem2]. (iii) \rightarrow (i) is trivial. (i) \rightarrow (ii): Since $\hat{X} = \{\{x\} \mid x \in X\} \subset \mathcal{F}(X)$ is homeomorphic to X , X has the MN operator G_X for X . Let H be any closed subset of X . First, using $L(x)$ we construct a sequence $\{S(H, n; x) \mid n \in \mathbb{N}\}$ of open neighborhoods of H in X with some property. Set

$$\mathcal{A}(H) = \{\{p, x_n\} \mid p \in H \setminus L(x), n \in \mathbb{N}\},$$

$$\mathcal{B}(H) = \{\{p, x\} \mid p \in X \setminus (H \cup L(x))\}.$$

Then $\mathcal{A}(H), \mathcal{B}(H)$ are separated in $\mathcal{F}(X)$. Let G be the MN operator for $\mathcal{F}(X)$. Since $G(\mathcal{A}(H), \mathcal{B}(H))$ is an open neighborhood of $\mathcal{A}(H)$ in $\mathcal{F}(X)$, for each $\{p, x_n\} \in \mathcal{A}(H)$ there exists the maximal open neighborhood $V(p; x_n, H)$ of p in X satisfying the following (1), (2) and (3):

$$(1) \quad \langle V(p; x_n, H), \{x_n\} \rangle \subset G(\mathcal{A}(H), \mathcal{B}(H));$$

$$(2) \quad \text{if } x_n \notin H, \text{ then}$$

$$V(p; x_n, H) \subset G_X(H, \{x_n\});$$

$$(3) \quad \text{if } x \notin H, \text{ then}$$

$$V(p; x_n, H) \subset G_X(H, \{x\}).$$

Set

$$S(H, n; x) = \bigcup \{V(p; x_n, H) \mid \{p, x_n\} \in \mathcal{A}(H)\}.$$

Then $\{S(H, n; x) | n \in \mathbb{N}, H \text{ closed in } X\}$ satisfies the following (4), (5) and (6):

(4) $S(H, n; x)$ is an open subset of X such that $H \setminus L(x) \subset S(H, n; x)$ for each n ;

(5) if $p \notin H$, then there exists $n \in \mathbb{N}$ such that $p \notin S(H, n; x)^-$.

To see (5), let $p \notin H$. If $p \notin L(x)$, then $\{p, x\} \in \mathcal{B}(H)$, which implies $\{p, x\} \notin G(\mathcal{A}(H), \mathcal{B}(H))^-$. Therefore there exist open neighborhoods $V(p)$, $V(x)$ of p , x in X , respectively, such that

$$\langle V(p), V(x) \rangle \cap G(\mathcal{A}(H), \mathcal{B}(H)) = \emptyset.$$

Take $x_n \in V(x)$. Then it is easy to see that $V(p) \cap S(H, n; x) = \emptyset$. If $p = x_n$ or $p = x$, then by (2) or (3),

$$V(p) = X \setminus G_X(H, \{x_n\})^- \text{ or } X \setminus G_X(H, \{x\})^-$$

satisfies $V(p) \cap S(H, n; x) = \emptyset$.

(6) If H, H' are closed subsets of X such that $H \subset H'$, then

$$S(H, n; x) \subset S(H', n; x) \text{ for each } n.$$

To see (6), observe that $\mathcal{A}(H) \subset \mathcal{A}(H')$ and $\mathcal{B}(H') \subset \mathcal{B}(H)$ hold. This means

$$(7) \quad G(\mathcal{A}(H), \mathcal{B}(H)) \subset G(\mathcal{A}(H'), \mathcal{B}(H')).$$

If $x_n \notin H'$ or $x \notin H'$, then

$$(8) \quad G_X(H, \{x_n\}) \subset G_X(H', \{x_n\}) \text{ or } G_X(H, \{x\}) \subset G_X(H', \{x\}) \text{ respectively.}$$

From (7) and (8) and the maximality of $V(p; x_n, H)$, $V(p; x_n, H) \subset V(p; x_n, H')$ follows. These prove (6).

Again, using $L(y)$ in place of $L(x)$ above, we can construct $\{S(H, n; y) | n \in \mathbb{N} \text{ and } H \text{ closed in } X\}$ satisfying the following (9), (10) and (11):

(9) $S(H, n; y)$ is an open subset of X such that $H \setminus L(y) \subset S(H, n; y)$ for each n ;

(10) if $p \notin H$, then there exists $n \in \mathbb{N}$ such that $p \notin S(H, n; y)^-$;

(11) if H, H' are closed subsets of X such that $H \subset H'$, then

$$S(H, n; y) \subset S(H', n; y) \text{ for each } n.$$

We define the stratification S by the following: Let H be any closed subset of X and let $f : \mathbb{N} \rightarrow \mathbb{N}^2$ be a bijection. For $n \in \mathbb{N}$, we define

$$S(H, n) = S(H, i; x) \cup S(H, j; y),$$

where $f(n) = (i, j)$. Then by (4), (5), (6) and (9), (10), (11) it is easily checked that S is the stratification of X , proving that X is stratifiable. \square

Corollary 2.1.2. *Let X be a space and let $Z(X)$ be the topological disjoint sum of X and two copies of $\{0\} \cup \{1/n | n \in \mathbb{N}\}$. Then $\mathcal{F}(Z(X))$ is MN if and only if X is stratifiable.*

The condition of existence of $L(x)$ and $L(y)$ in the above necessary. In fact, monotone normality of $\mathcal{F}(X)$ does not mean stratifiability of X as shown by the next example:

Example 2.1.3. There exists a non-stratifiable space X such that $\mathcal{F}(X)$ is MN.

Construction. As X , we take the same space $X = [0, \omega_1]$ as in [HLZ73, Example7.6], which is topologized in such a way that all $\alpha < \omega_1$ are isolated and basic neighborhoods of ω_1 are sets of the form $(\alpha, \omega_1]$. Then, as shown there, for each $n \in \mathbb{N}$ X^n is MN but X is not stratifiable. We show that $\mathcal{F}(X)$ is MN. Since the mapping $f : X^n \rightarrow \mathcal{F}_n(X) = \{F \in \mathcal{F}(X) \mid |F| \leq n\}$ with the relative topology, define by

$$f((x_1, \dots, x_n)) = \{x_1, \dots, x_n\} \text{ for } (x_1, \dots, x_n) \in X^n$$

is perfect, $\mathcal{F}_n(X)$ is MN by [HLZ73, Theorem2.6]. Let Z be the topological disjoint sum of $\{\mathcal{F}_n(X) | n \in \mathbb{N}\}$, which is also MN. Let $g : Z \rightarrow \mathcal{F}(X)$ be the natural mapping. Then it is easy to see that g is a closed mapping. Hence by [HLZ73, Theorem2.6] is MN. \square

Moreover, for the same space X as above, $\mathcal{K}(X) = \mathcal{F}(X)$ holds. Therefore MN and stratifiability of $\mathcal{K}(X)$ do not coincide. On the other hand, MN of X does not mean MN of $\mathcal{K}(X)$ because there exists a stratifiable space X such that $\mathcal{K}(X)$ is not normal [Bor80].

Theorem 2.1.4. *The following statements are true:*

(i) *For a space, if $\mathcal{K}(Y^{\omega_0})$ is MN, then $\mathcal{K}(Y)$ is stratifiable.*

(ii) *Let X be a space with $L(x)$. If $\mathcal{K}(X^2)$ is MN, then $\mathcal{K}(X)$ is stratifiable.*

Proof. (i): It suffices to show that $\mathcal{K}(Y)^{\omega_0}$ is MN [HLZ73, Corollary4.2]. To see it, let $f : \mathcal{K}(Y)^{\omega_0} \rightarrow \mathcal{K}(Y^{\omega_0})$ be defined by

$$f((K_1, K_2, \dots)) = K_1 \times K_2 \times \dots \text{ for } (K_1, K_2, \dots) \in \mathcal{K}(Y)^{\omega_0}.$$

Then we can show easily that f is the embedding. Since MN is hereditary, $\mathcal{K}(Y)^{\omega_0}$ is MN. (ii): It suffices to show that $\mathcal{K}(X) \times L(x)$ is MN [HLZ73, Theorem4.1]. To see it, we define a mapping $g : \mathcal{K}(X) \times L(x) \rightarrow \mathcal{K}(X^2)$ by

$$g((K, p)) = K \times \{p\} \text{ for } (K, p) \in \mathcal{K}(X) \times L(x).$$

Then it is easy to see that g is the embedding. Hence $\mathcal{K}(X) \times L(x)$ is MN. \square

2.1.3 Monotone normality of mapping spaces.

Let $C(X, Y)$ be a set of all continuous mapping of X to Y . As its topology, compact-open topology is well-known. We write by $C_K(X, Y)$ the space with this topology which has a base consisting of the subsets of the form:

$$W(K_1, \dots, K_n; U_1, \dots, U_n) = \{f \in C(X, Y) | f(K_i) \subset U_i \text{ for each } i\},$$

where $K_1, \dots, K_n \in \mathcal{K}(X)$ and $U_1, \dots, U_n \in \tau_Y$. In particular, if all K_i are singletons, then the topology is called the pointwise convergence topology and the space $C(X, Y)$ with this topology is written by $C_p(X, Y)$. We define a subclass

$M^*(\mathcal{K}(X), \mathcal{K}(Y))$ of $C(\mathcal{K}(X), \mathcal{K}(Y))$ as follows: $f \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$ if and only if $f \in C(\mathcal{K}(X), \mathcal{K}(Y))$ and f satisfies the following three conditions (i) f is *monotone*, i.e., $f(K) \subset f(K')$ if $K, K' \in \mathcal{K}(X)$ and $K \subset K'$, (ii) f is *finitely additive* for $\mathcal{F}(X)$, i.e., $f(F_1 \cup F_2) = f(F_1) \cup f(F_2)$ if $F_1, F_2 \in \mathcal{F}(X)$, (iii) if $x \in X$, then $|f(\{x\})| = 1$.

Proposition 2.1.5. $C_K(X, Y) \cong M_p^*(\mathcal{K}(X), \mathcal{K}(Y))$.

Proof. Define a mapping $\varphi : C_K(X, Y) \rightarrow M_p^*(\mathcal{K}(X), \mathcal{K}(Y))$ by

$$\varphi(f)(K) = f(K) \text{ for any } K \in \mathcal{K}(X) \text{ and for any } f \in C_K(X, Y).$$

Then $\varphi(f) \in C(\mathcal{K}(X), \mathcal{K}(Y))$ [Mic51, Theorem5.10.1]. Easily $\varphi(f) \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$ follows. Obviously φ is one-to-one. To see the continuity of φ , let

$$\varphi(f) \in W(\{K\}; \langle U_1, \dots, U_k \rangle),$$

where $U_1, \dots, U_k \in \tau_Y$ and $K \in \mathcal{K}(X)$. Then $f(K) \in \langle U_1, \dots, U_k \rangle$, which means $f(K) \subset \bigcup_{i=1}^k U_i$ and $f(K) \cap U_i \neq \emptyset$ for each i . Take $x_i \in K$ such that

$$p_i = f(x_i) \in f(K) \cap U_i, \quad i = 1, \dots, k.$$

Let

$$\hat{O} = W\left(K, \bigcup_{i=1}^k U_i\right) \cap \bigcap_{i=1}^k W(\{x_i\}, U_i).$$

Then \hat{O} is an open neighborhood of f in $C_K(X, Y)$ and easily we have

$$\varphi(\hat{O}) \subset W(\{K\}; \langle U_1, \dots, U_k \rangle).$$

To see that φ is open, let $W(K_1, \dots, K_n; U_1, \dots, U_n)$ be any basic open subset of $C_K(X, Y)$. Let $f \in W(K_1, \dots, K_n; U_1, \dots, U_n)$. Then $f(K_i) \in \langle U_i \rangle$ for each i , which means

$$\varphi(f) \in W(\{K_1\}, \dots, \{K_n\}; \langle U_1 \rangle, \dots, \langle U_n \rangle).$$

Moreover, easily we have

$$W(\{K_1\}, \dots, \{K_n\}; \langle U_1 \rangle, \dots, \langle U_n \rangle) \cap \varphi(C(X, Y))$$

$$\subset \varphi(W(K_1, \dots, K_n; U_1, \dots, U_n)),$$

proving that $\varphi(W(K_1, \dots, K_n; U_1, \dots, U_n))$ is open in $M_p^*(\mathcal{K}(X), \mathcal{K}(Y))$. Thus it remains to show that φ is onto. Let $F \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$. Then $f = F|_{\hat{X}} \in C(X, Y)$, where $\hat{X} = \mathcal{F}_1(X) \subset \mathcal{K}(X)$. Assume $F \neq \varphi(f)$. That is, there exists $K \in \mathcal{K}(X)$ such that $\varphi(f)(K) \neq F(K)$. Since $F \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$, $f(K) \subset F(K)$, which means that there exists a point $y \in F(K) \setminus f(K)$. Take disjoint open neighborhoods U, V of $y, f(K)$ in Y , respectively. Since F is continuous, there exists an open neighborhood $\langle O_1, \dots, O_t \rangle$ of K such that $F(\langle O_1, \dots, O_t \rangle) \subset \langle U, V, Y \rangle$. Take $p_i \in K \cap O_i$ for each i . Then by $F \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$,

$$F(\{p_1, \dots, p_t\}) = f(\{p_1, \dots, p_t\}).$$

By (i) there, $F(\{p_1, \dots, p_t\}) \subset f(K)$. But this is a contradiction to $F(\{p_1, \dots, p_t\}) \in \langle U, V, Y \rangle$. \square

Lemma 2.1.6. *If a space X has a pseudobase \mathcal{K}_0 such that $\mathcal{K}_0 \subset \mathcal{K}(X)$, then*

$$M_p^*(\mathcal{K}(X), \mathcal{K}(Y)) \hookrightarrow M_p^*(\mathcal{K}_0, \mathcal{K}(Y)).$$

Proof. Note that \mathcal{K}_0 is a dense subset of $\mathcal{K}(X)$. So, if we define

$$\varphi : M_p^*(\mathcal{K}(X), \mathcal{K}(Y)) \rightarrow M_p^*(\mathcal{K}_0, \mathcal{K}(Y)) \text{ by } \varphi(f) = f|_{\mathcal{K}_0}$$

for each $f \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$, then φ is one-to-one and $\varphi(f) \in M^*(\mathcal{K}_0, \mathcal{K}(Y))$. To see that φ is open, let

$$W^* = W(\{K_1\}, \dots, \{K_s\}; \hat{U}_1, \dots, \hat{U}_s)$$

be any basic open subset of $M_p^*(\mathcal{K}(X), \mathcal{K}(Y))$, where for each i $K_i \in \mathcal{K}(X)$ and

$$\hat{U}_j = \langle U_{j1}, \dots, U_{jk(j)} \rangle$$

with U_{ji} open in X . Let f_0 be any element of $\varphi(W^*)$. Then $f_0 = \varphi(f)$, where $f \in W^*$. Let j be fixed for a while. $f(K_j) \in \hat{U}_j$ implies that there exists an

open neighborhood $\hat{V}_j = \langle V_{j1}, \dots, V_{jt(j)} \rangle$ of K_j such that $f(\hat{V}_j) \subset \hat{U}_j$. Take $p_{ji} \in K_j \cap V_{ji}$, $i = 1, \dots, t(j)$ and let $F = \{p_{j1}, \dots, p_{jt(j)}\}$. Then $f(F) \in \hat{U}_j$. Since $f \in M^*(\mathcal{K}(X), \mathcal{K}(Y))$, $f(F) = \{f(p_{j1}), \dots, f(p_{jt(j)})\}$ and $|f(p_{ji})| = 1$. Since $f(F) \in \hat{U}_j$ for each $i = 1, \dots, t(j)$ there exists $s(i) \in \{1, \dots, k(j)\}$ such that $f(p_{ji}) \in U_{js(i)}$ and $\{s(i) | i = 1, \dots, t(j)\} = \{1, \dots, k(j)\}$. Since \mathcal{K}_0 is a pseudobase for X , for each i there exists $L_i \in \mathcal{K}_0$ such that $p_{ji} \in L_i \subset V_{ji}$ and $f(L_i) \in \langle U_{js(i)} \rangle$. Also, there exists $L_0 \in \mathcal{K}_0$ such that

$$K \cup \bigcup_{i=1}^{t(j)} L_i \subset L_0 \subset \bigcup_{i=1}^{t(j)} V_{ji}.$$

Let

$$\hat{O}(j) = \bigcap_{i=1}^{t(j)} W(\{L_i\}; \langle U_{js(i)} \rangle) \cap W(\{L_0\}; \hat{U}_j).$$

Then $\hat{O}(j)$ is an open neighborhood of f_0 in $M_p^*(\mathcal{K}_0, \mathcal{K}(Y))$. It is easy to see $\hat{O} = \bigcap_{j=1}^s \hat{O}(j)$ is an open neighborhood of f_0 such that

$$\hat{O} \cap \varphi(M_p^*(\mathcal{K}(X), \mathcal{K}(Y))) \subset \varphi(W^*),$$

proving that φ is open. □

Theorem 2.1.7. *Let X be a compact metric space and let $\mathcal{K}(Y)$ be semistratifiable. Then $C_K(X, Y)$ is MN if and only if $C_K(X, Y)$ is stratifiable.*

Proof. Note that X has a countable pseudobase \mathcal{K}_0 such that $\mathcal{K}_0 \subset \mathcal{K}(X)$. By lemmas,

$$C_K(X, Y) \hookrightarrow \mathcal{K}(Y)^{\omega_0}.$$

Since $\mathcal{K}(Y)^{\omega_0}$ is semistratifiable, being MN and stratifiable coincide by [HLZ73, Theorem2.5]. □

We remark that even if $X = [0, 1]$ and Y is stratifiable, $C_K(X, Y)$ need not be normal [Mic66, Example12.1].

2.2 Near Metric Properties of Hyperspaces

Near metric properties of the hyperspace of closed, compact and finite subsets of a space X are examined. In particular, the properties of monotone normality and stratifiability are investigated.

2.2.1 Introduction

This section aims to examine when a hyperspace of a topological space X possesses certain general metric properties. Given a space X , which we will henceforth assume to be Tychonoff, we define a hyperspace of X to be one of the following:

$$\mathcal{H}(X) = \{A \subseteq X : A \text{ is non-empty and closed in } X\},$$

$$\mathcal{K}(X) = \{A \in \mathcal{H}(X) : A \text{ is compact}\} \text{ and}$$

$$\mathcal{F}(X) = \{A \in \mathcal{H}(X) : A \text{ is finite}\}.$$

The set $\mathcal{H}(X)$ is given the Vietoris topology, that is, the topology generated by sets of the form $[U_1 \dots U_n] = \{A \in \mathcal{H}(X) : A \subseteq \bigcup U_i \text{ and } A \cap U_i \neq \emptyset\}$, where each U_i is a non-empty open subset of X . The subsets $\mathcal{K}(X)$ and $\mathcal{F}(X)$ of $\mathcal{H}(X)$ are given the subspace topology. The map $x \mapsto \{x\}$ embeds X as a closed subspace of $\mathcal{H}(X)$.

We determine when the hyperspaces $\mathcal{H}(X), \mathcal{K}(X), \mathcal{F}(X)$ are monotonically normal, stratifiable or cosmic (definitions are given below). The case of $\mathcal{H}(X)$ is brief— X must be compact and metrisable. On the other hand, properties of $\mathcal{F}(X)$ will be seen to mirror those of (finite powers of) the space X . The situation for $\mathcal{K}(X)$ is murkier. We essentially show that $\mathcal{K}(X)$ is monotonically normal if and only if either $\mathcal{K}(X) = \mathcal{F}(X)$ or $\mathcal{K}(X)$ is stratifiable. In addition, we give a consistent and independent criterion for $\mathcal{K}(X)$ to be cosmic. From this follows a consistent and independent criterion for a space to be separable metrisable. To achieve this we additionally present some results on function spaces in the compact-open and the pointwise topologies.

A space X is monotonically normal (MN) if, for each pair (A, B) of disjoint closed subsets of X , there is an open subset $H(A, B)$ of X such that

- (i) $A \subseteq H(A, B) \subseteq \overline{H(A, B)} \subseteq X \setminus B$
- (ii) $A \subseteq A'$ and $B' \subseteq B$ implies $H(A, B) \subseteq H(A', B')$.

It will be convenient, however, to use a “local” characterisation of MN:

for every point x and open neighbourhood U of x , there is an open neighbourhood $V(x, U)$ of x such that $V(x, U) \cap V(x', U') \neq \emptyset \implies x \in U'$ or $x' \in U$.

Without loss of generality, we can assume that $V(x, U) \subseteq U$ and that $V(x, U) \subseteq V(x, U')$ for $U \subseteq U'$. It is sufficient for V to be defined for members of a base for X . Monotone normality is a hereditary property and is preserved by closed maps.

Stratifiability can be thought of as “monotone perfect normality”. A space X is stratifiable if, for each closed subset A of X , and natural number n , there is an open subset $G(A, n)$ of X containing A such that

- (i) $G(A, m) \subseteq G(B, n)$ whenever $A \subseteq B$ and $m \geq n$
- (ii) $A = \bigcap G(A, n)$, and
- (iii) $A = \bigcap \overline{G(A, n)}$.

A space is stratifiable if and only if it is both monotonically normal and a σ -space (possesses a σ -discrete network). Recall that a space is cosmic if it possesses a countable network. Then a space is separable and stratifiable if and only if it is both monotonically normal and cosmic. A result of Heath’s, which we shall use in the sequel, states that if S is a space with a countable limit point, then $X \times S$ is MN only if X is stratifiable.

The reader is referred to [Gru84] for a survey of these and other generalised metric properties. References for all the facts quoted above will be found there.

2.2.2 The Space of Closed Subsets

An instance of a theorem of Fedorchuk (Theorem 4.20) in [HH92] states that a compact Hausdorff space X is metrisable if and only if $\mathcal{H}(X)$ is hereditarily normal. Since normality of $\mathcal{H}(X)$ implies the compactness of X , we have the result proved

directly in [BM]:

Theorem 2.2.1 (Brandma and van Mill). *The hyperspace $\mathcal{H}(X)$ is monotonically normal if and only if X is compact metrisable.*

For the same reason, $\mathcal{H}(X)$ is cosmic or stratifiable if and only if X is compact metrisable. Thus, in this respect, $\mathcal{H}(X)$ is “too large” to possess interesting near metric properties without it being compact and metrisable.

2.2.3 The Space of Finite Subsets

In contrast to the case for $\mathcal{H}(X)$, properties of $\mathcal{F}(X)$ are much closer to those of X .

We will use two main facts:

- (1) that $\mathcal{F}(X)$ is the countable union of the closed subspaces $\mathcal{F}_n(X)$, for $n \in \omega$, where $\mathcal{F}_n(X) = \{A \in \mathcal{F}(X) : |A| \leq n\}$.
- (2) that the mapping $\pi_n : X^n \rightarrow \mathcal{F}_n(X)$ is a closed, continuous, surjection (and therefore transfers a “point and open set” MN operator on X^n to one on $\mathcal{F}_n(X)$ in the standard way).

Theorem 2.2.2. *Let X be a space. Then the following are equivalent:*

- (1) X^2 is MN
- (2) X^n is MN for all $n \in \omega$
- (3) $\mathcal{F}(X)$ is MN

Proof. The equivalence of (1) and (2) is shown by Gartside in [Gar93], who proves the more general result that the product of a finite collection of spaces is MN, if the product of any pair is MN. In the above case, let us suppose that X^2 has MN operator V^2 , and write

$$V^2((x, y), U_x \times U_y) = V_x^2((x, y), U_x \times U_y) \times V_y^2((x, y), U_x \times U_y).$$

We may suppose that V^2 is symmetric in the sense that $V_x^2((x, y), U_x \times U_y) = V_x^2((y, x), U_y \times U_x)$. (This can be achieved by re-defining $V^2((x, y), U_x \times U_y)$ as

$$V^2((x, y), U_x \times U_y) \cap V^2((y, x), U_y \times U_x)^{-1}.$$

Then $V^n((x_1 \dots x_n), U_1 \times \dots \times U_n)$

$$= \bigcap_{i=1}^n V_{x_1}^2((x_1, x_i), U_1 \times U_i) \times \dots \times \bigcap_{i=1}^n V_{x_n}^2((x_n, x_i), U_n \times U_i)$$

is an MN operator on X^n , as required.

(1) \implies (3) By the above, we have an MN operator V^n , defined on each X^n , and by fact (2), each $\mathcal{F}_n(X)$ is therefore MN with an MN operator V_n . We define what we shall show is an MN operator on $\mathcal{F}(X)$ by

$$V_\omega(\{x_1 \dots x_n\}, [U_1 \dots U_n]) = \left[\bigcap_{i=1}^n V_{x_1}^2((x_1, x_i), U_1 \times U_i), \dots, \bigcap_{i=1}^n V_{x_n}^2((x_n, x_i), U_n \times U_i) \right]$$

where $[U_1 \dots U_n]$ is a basic open set containing $\{x_1 \dots x_n\}$, and the U_i are pairwise disjoint with $x_i \in U_i$.

We prove that V_ω is an MN operator by comparing V_n and the restriction of V_ω to $\mathcal{F}_n(X)$; specifically, we check that, for $k \leq n$,

$$V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap \mathcal{F}_n(X) \subseteq V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k]). \quad (*)$$

For suppose that (*) holds, and that

$$V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap V_\omega(\{y_1 \dots y_m\}, [W_1 \dots W_m]) \neq \emptyset.$$

Then this non-empty intersection is witnessed by a point in $\mathcal{F}_n(X)$, where $n \geq \max(k, m)$ (the inequality can be strict). By (*),

$$V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap V_n(\{y_1 \dots y_m\}, [W_1 \dots W_m]) \neq \emptyset.$$

Since we know that V_n is an MN operator on $\mathcal{F}_n(X)$, either

$$\{x_1 \dots x_k\} \in [W_1 \dots W_m] \quad \text{or} \quad \{y_1 \dots y_m\} \in [U_1 \dots U_k].$$

Hence V_ω is indeed an MN operator on $\mathcal{F}(X)$.

So, it remains to check that (*) holds. Now $V_n(\{x_1 \dots x_k\}, [U_1 \dots U_k])$ is defined to be $\pi_n^\# \left(\bigcup_{(a_1 \dots a_n) \in \pi_n^{-1}(\{x_1 \dots x_n\})} V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_n])) \right)$ (where $\pi_n^\#(S) = \{F \in \mathcal{F}_n(X) : \pi_n^{-1}F \subseteq S\}$). Let $\{z_1 \dots z_m\} \in V_\omega(\{x_1 \dots x_k\}, [U_1 \dots U_k]) \cap \mathcal{F}_n(X)$ (so

$k \leq m \leq n$). Let $(b_1 \dots b_n) \in \pi_n^{-1}(\{z_1 \dots z_k\})$ (so every b is some z , and every z is some b).

We want to show that $(b_1 \dots b_n) \in V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_n]))$ for some $(a_1 \dots a_n) \in \pi_n^{-1}(\{x_1 \dots x_n\})$.

Each z is in one set of the form $\bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$ (these sets are disjoint, since the U_i were assumed to be), and for every j , $1 \leq j \leq k$, some z (possibly more than one) is in $\bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$.

Choose a_l to be the x_j such that $b_l \in \bigcap_{i=1}^k V_{x_j}^2((x_j, x_i), U_j \times U_i)$ for $1 \leq l \leq n$. Then $\{a_1 \dots a_n\} = \{x_1 \dots x_k\}$, so $(a_1 \dots a_n) \in \pi_n^{-1}(\{x_1 \dots x_k\})$ as required.

Denoting by p_{a_j} the index such that $a_j = x_{p_{a_j}}$, we have that

$$b_j \in \bigcap_{i=1}^k V_{a_j}^2((a_j, a_i), U_{p_{a_j}} \times U_{p_{a_i}}) = \bigcap_{i=1}^n V_{a_j}^2((a_j, a_i), U_{p_{a_j}} \times U_{p_{a_i}}).$$

Finally, looking at the definition of V^n in terms of V^2 , we can see that $(b_1 \dots b_n) \in V^n((a_1 \dots a_n), U_{p_{a_1}} \times \dots \times U_{p_{a_n}}) \subseteq V^n((a_1 \dots a_n), \pi_n^{-1}([U_1 \dots U_k]))$. Thus we have proved (*), and the proof is complete.

(3) \implies (1) We will use only the fact that $\mathcal{F}_2(X)$ has an MN operator V . Write, for $x \neq y$,

$$V(\{x, y\}, [U_x, U_y]) = \left[V_x(\{x, y\}, [U_x, U_y]), V_y(\{x, y\}, [U_x, U_y]) \right],$$

where $x \in V_x(\{x, y\}, [U_x, U_y]) \subseteq U_x$ and similarly for y ; and

$$V(\{x\}, [U_x]) = [V(x, U_x)].$$

Then define V^2 by

$$V^2((x, y), U_x \times U_y) = \left(V_x(\{x, y\}, [U_x, U_y]) \cap V(x, U_x) \right) \times \left(V_y(\{x, y\}, [U_x, U_y]) \cap V(y, U_y) \right),$$

where U_x and U_y are disjoint if $x \neq y$, and equal if $x = y$. It can easily be checked that V^2 is an MN operator on X^2 . \square

Following immediately from this is a result of [MK87] :

Corollary 2.2.3 (Mizokami and Koiwa). *X is stratifiable if and only if $\mathcal{F}(X)$ is stratifiable.*

Proof. The reverse implication is obvious. For the direct implication, let X be stratifiable. Then every finite power of X is stratifiable and therefore MN, thus $\mathcal{F}(X)$ is MN. Moreover, each finite power of X is, since stratifiable, a σ -space. Then $\mathcal{F}(X)$ is the countable union of closed σ -spaces, and therefore a σ -space. Hence $\mathcal{F}(X)$ is stratifiable. \square

Corollary 2.2.4. *If $\mathcal{F}(X)$ is MN then $\mathcal{F}(X)^n$ is MN $\forall n \in \omega$*

Proof. Observe that $\mathcal{F}(X)^n$ embeds in $\mathcal{F}(X \oplus \dots \oplus X)$. If $\mathcal{F}(X)$ is MN, then X^2 is MN by 2.2.2, and therefore so is $(X \oplus \dots \oplus X)^2$. By 2.2.2 again, $\mathcal{F}(X \oplus \dots \oplus X)$ is MN, and hence $\mathcal{F}(X)^n$ is MN. \square

For comparison we have the following example [Gar93]:

Example 2.2.5. There is a space X such that all finite powers of X are MN, but it is not linearly stratifiable. Thus $\mathcal{F}(X)$ is MN but not (linearly) stratifiable.

(A space is κ -stratifiable if it is stratifiable as defined above, but the open sets are indexed by a cardinal κ instead of ω ; and linearly stratifiable if it is κ -stratifiable for some infinite cardinal κ . Every linearly stratifiable space is MN.)

2.2.4 The Space of Compact Subsets

If a space X contains no infinite compact subsets, then $\mathcal{K}(X) = \mathcal{F}(X)$, and we are back in the situation of the preceding subsection. The first result of this subsection says that if $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then X is stratifiable. This suggests that if $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then $\mathcal{K}(X)$ should be stratifiable. The second and third results of this subsection show that this conjecture is true under additional assumptions.

Proposition 2.2.6. *If $\mathcal{K}(X) \neq \mathcal{F}(X)$ and $\mathcal{K}(X)$ is MN, then X is stratifiable.*

Proof. Since $\mathcal{K}(X)$ is MN, so too is $\mathcal{F}(X)$, and all finite powers of X are MN. As $\mathcal{K}(X) \neq \mathcal{F}(X)$, there is a countably infinite non-discrete subset of X . Now $X \times S$ is MN (as a subspace of X^2), and X is stratifiable by Heath's result. \square

Call a space X non-trivial if every non-empty open subset of X contains an infinite compact subset. When $\mathcal{K}(X)$ is MN, as stratifiable compact spaces are metrisable, this is equivalent to saying: every non-empty open subset of X contains a convergent sequence, $x_n \rightarrow x_\omega$, such that $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega + 1$.

Theorem 2.2.7. *Let X be non-trivial, and $\mathcal{K}(X)$ MN. Then $\mathcal{K}(X)$ is stratifiable.*

Proof. Let us observe first that if X is compact, then $\mathcal{K}(X) = \mathcal{H}(X)$, and thus is compact and metrisable. Otherwise it will suffice to show that $\mathcal{K}(X)$ is a σ -space. If X is not compact, then, since X is stratifiable, there exists an infinite discrete family $\{U_n : n \in \omega\}$ of non-empty open subsets of X . For $n \in \omega$, define $\mathcal{K}_n = \{K \in \mathcal{K}(X) : K \cap U_n = \emptyset\}$.

Then, as the complement of the basic open set $[X, U_n]$ in $\mathcal{K}(X)$, each \mathcal{K}_n is closed. Also, $\mathcal{K}(X) = \bigcup_n \mathcal{K}_n$ by the discreteness of $\{U_n : n \in \omega\}$.

Next we show that, for each n , $\mathcal{K}_n \times (\omega + 1)$ embeds in $\mathcal{K}(X)$. By non-triviality of X , pick a sequence $(x_\alpha^n)_{\alpha < \omega + 1}$ contained in U_n . Then it is straightforward to check that $A \times \{\alpha\} \mapsto A \cup \{x_\alpha^n\}$ defines an embedding of $\mathcal{K}_n \times (\omega + 1)$ into $\mathcal{K}(X)$.

Thus $\mathcal{K}_n \times (\omega + 1)$ is MN, hence \mathcal{K}_n is stratifiable, and therefore is a σ -space. Thus $\mathcal{K}(X)$, as a countable union of closed σ -spaces, is a σ -space, as required. \square

Note that the proof shows that, if $\mathcal{K}(X)$ is MN and X contains an infinite discrete family of open sets, each containing a convergent sequence, then $\mathcal{K}(X)$ is stratifiable.

For the next result, in the separable case, recall that space is \aleph_0 if it possesses a countable family \mathcal{N} of subsets of X such that for every compact subset K of X contained in an open set U , there exists an $N \in \mathcal{N}$ such that $K \subseteq N \subseteq U$. It is a well-known result that

Proposition 2.2.8. *$\mathcal{K}(X)$ is cosmic if and only if X is \aleph_0 .*

Theorem 2.2.9. *Let $\mathcal{K}(X)$ be separable and MN, and suppose that X contains two disjoint convergent sequences S_1 and S_2 . Then $\mathcal{K}(X)$ is stratifiable.*

Proof. It is sufficient to show that $\mathcal{K}(X)$ is cosmic. Since the S_i are disjoint compact sets in X , they are separated by open sets U_i . Define $\mathcal{K}_i = \mathcal{K}(X \setminus U_i)$. Then $\mathcal{K}_i \times (\omega + 1)$ embeds in $\mathcal{K}(X)$ as above, thus \mathcal{K}_i is separable and stratifiable. Hence it is \aleph_0 , and so $X \setminus U_i$ is \aleph_0 . Then $X = (X \setminus U_1) \cup (X \setminus U_2)$ is the union of two closed \aleph_0 subspaces and hence \aleph_0 . Thus $\mathcal{K}(X)$ is cosmic. \square

The following non-monotone version of Theorem 2.2.7 follows the lines of the classic proof by Katětov [Kat48]:

Theorem 2.2.10. *Let X be non-trivial, and $\mathcal{K}(X)$ hereditarily normal. Then points of $\mathcal{K}(X)$ are G_δ .*

Proof. Pick $A \in \mathcal{K}(X)$ and suppose that $A \neq X$ (if $A = X$ then $\mathcal{K}(X)$ is compact and metrisable). Then, since X is normal, there exists a non-empty open subset of X with closure disjoint from A . Inside this closure, pick a non-trivial convergent sequence, $x_n \rightarrow x_\omega$, with $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega + 1$. Let $S = \{x_\alpha : \alpha < \omega + 1\}$.

Define $\mathcal{P} = \{A \cup F : F \subseteq [S \setminus \{x_\omega\}]^{<\omega}\}$ and $\mathcal{Q} = \{K \in \mathcal{K}(X) : K \cap S = \{x_\omega\} \text{ and } K \setminus \{x_\omega\} \neq A\}$.

Then $C \in \overline{\mathcal{Q}}$ implies $\{x_\omega\} \in C$ so $\mathcal{P} \cap \overline{\mathcal{Q}} = \emptyset$. Also, $C \in \overline{\mathcal{P}}$ implies $C = A \cup I$ for some $I \subseteq S$, so $\overline{\mathcal{P}} \cap \mathcal{Q} = \emptyset$.

So by hereditary normality, there exists an open set \mathcal{G} in $\mathcal{K}(X)$ s.t. $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{G} \cap \overline{\mathcal{Q}} = \emptyset$.

For $\mathcal{U} = [U_1 \dots U_m]$, with $A \cup \{x_n\} \in \mathcal{U} \subseteq \mathcal{G}$, satisfying $x_n \in U_m$, $U_m \cap A = \emptyset$, and $x_n \notin U_i$ for $i \neq m$, define $\mathcal{U}^{-n} = [U_1 \dots U_{m-1}]$. Then $A \in \mathcal{U}^{-n}$, and also $B \in \mathcal{U}^{-n} \implies B \cup \{x_n\} \in \mathcal{U}$, and $x_n \notin B$. For $n \in \omega$, define $\mathcal{G}_n = \bigcup \{\mathcal{U}^{-n} : \mathcal{U} \text{ as above}\}$. Then \mathcal{G}_n is open and contains A .

Claim: $\{A\} = \bigcap_n \mathcal{G}_n \cap (\mathcal{K}(X) \setminus \{A \cup \{x_\omega\}\})$

Proof of the claim. Suppose $B \in \mathcal{G}_n \cap (\mathcal{K}(X) \setminus \{A \cup \{x_\omega\}\})$, $B \neq A$. Then $\forall n \in \omega$, $B \in \mathcal{G}_n$ so $B \cup \{x_n\} \in \mathcal{G}$, and $x_n \notin B$. Then $B \cup \{x_\omega\} \in \overline{\mathcal{G}}$. But $B \cup \{x_\omega\} \cap S = \{x_\omega\}$ and $(B \cup \{x_\omega\}) \setminus \{x_\omega\} \neq A$ so $B \cup \{x_\omega\} \in \mathcal{Q}$ – contradicting $\overline{\mathcal{G}} \cap \mathcal{Q} = \emptyset$.

Hence points of $\mathcal{K}(X)$ are G_δ . □

2.2.5 Interlude—Function Spaces

Let X and Y be spaces, and write $C(X, Y)$ for the set of all continuous functions of X into Y . We abbreviate $C(X, \mathbb{R})$ by $C(X)$. For $A \subseteq X$ and $U \subseteq Y$, define $B(A, U) = \{f \in C(X, Y) : f[A] \subseteq U\}$. Letting A range over finite (respectively, compact) subsets of X and U range over open subsets of Y , the $B(A, U)$ form a subbase for the topology of pointwise convergence (respectively, compact-open topology). Write $C_p(X, Y)$ for $C(X, Y)$ with the topology of pointwise convergence, and $C_k(X, Y)$ for $C(X, Y)$ with the compact-open topology.

We explain how the space $C_k(X, Y)$ may be related to the spaces $C_p(X', Y')$, $\mathcal{K}(X)$ and $\mathcal{K}(Y)$. Since the topology of pointwise convergence has been intensively studied, this provides a useful tool for investigating the compact-open topology.

Let f be a continuous function of X to Y . Lift f to a continuous function $\mathcal{K}f : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ by defining $\mathcal{K}f(K) = f[K]$. This gives a map of $C_k(X, Y)$ into $C_p(\mathcal{K}(X), \mathcal{K}(Y))$. It is straightforward to check that this map is a (topological) embedding.

Proposition 2.2.11. *The map $f \mapsto \mathcal{K}f$ embeds $C_k(X, Y)$ into $C_p(\mathcal{K}(X), \mathcal{K}(Y))$.*

Let us apply Proposition 2.2.11 to cardinal invariants of $C_k(X)$. From Proposition 2.2.11, $C_k(X)(= C_k(X, \mathbb{R}))$ embeds in $C_p(\mathcal{K}(X), \mathcal{K}(\mathbb{R}))$. Now $\mathcal{K}(\mathbb{R})$ is a separable metric space, and so embeds in \mathbb{R}^ω ; also, $C_p(\mathcal{K}(X), \mathbb{R}^\omega) = C_p(\mathcal{K}(X))^\omega$. Hence, $C_k(X)^\omega$ embeds in $C_p(\mathcal{K}(X))^\omega$. The following result is a consequence of this last fact, and well-known results about $C_p(Z)$.

Corollary 2.2.12. *If, for all $n \in \omega$, $\mathcal{K}(X)^n$ is:*

- (1) Lindelöf, (2) hereditarily Lindelöf, (3) hereditarily separable,
- (4) hereditarily ccc,

then, for all $n \in \omega$, $C_k(X)^n$ is:

- (1) *countably tight*, (2) *hereditarily separable*, (3) *hereditarily Lindelöf*,
 (4) *hereditarily ccc (respectively)*.

Part (4) of the above result plays a vital role in the next subsection.

2.2.6 Cosmicity of $\mathcal{K}(X)$

This subsection examines when $\mathcal{K}(X)$ is cosmic, away from the monotone properties considered above. The following theorem requires the following extra-ZFC axiom:

The Open Colouring Axiom (OCA). *If $[X]^2 = K_0 \cup K_1$ is a given partition where $X \subseteq \mathbb{R}$ and where K_0 is open in $[X]^2$, then either there is an uncountable 0-homogeneous set, or else X is the union of countably many 1-homogeneous sets.*

For more details, and applications, of OCA, the reader is referred to [Tod89]. We shall be content to note here that OCA follows from PFA, but that ZFC and (ZFC + OCA) are equiconsistent.

Condition (CK) was defined by Gartside and Reznichenko in [GR]: A space X satisfies (CK) if there is a σ -compact subset Y of X such that for every compact subset K of X , there is a compact subset L satisfying $K \subseteq \overline{L \cap Y}$. It is used here through the following

Theorem 2.2.13 (Gartside and Reznichenko). *X has (CK) if and only if $C_k(X)$ is cometrisable.*

Above, a space Y is cometrisable if there is a coarser metric topology on Y , and for each point of Y a neighbourhood base of metric closed sets. Also for the following theorem and example, observe, as is well known, that a space Y has Y^ω hereditarily ccc if and only if Y^n is hereditarily ccc for all $n \in \omega$. (A space is hereditarily ccc if it does not contain any uncountable discrete subspaces.)

Theorem 2.2.14 (OCA).

- (i) *$\mathcal{K}(X)$ is cosmic if and only if $\mathcal{K}(X)^\omega$ is hereditarily ccc and X has (CK).*

(ii) $\mathcal{K}(X)$ is separable metrisable if and only if $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc.

Proof. Let us first suppose that $\mathcal{K}(X)$ is cosmic. Then X is \aleph_0 and therefore has (CK) as observed in [GR]. Also, cosmicity of $\mathcal{K}(X)$ implies cosmicity of $\mathcal{K}(X)^\omega$, and therefore $\mathcal{K}(X)^\omega$ is hereditarily ccc. Conversely, suppose that $\mathcal{K}(X)^\omega$ is hereditarily ccc and X has (CK). Then $C_k(X)$ is cometrisable, and, by Corollary 2.2.12, is hereditarily ccc in all of its finite powers. By a theorem of Gruenhage [Gru89], originally proved under PFA, but subsequently shown by Todorčević [Tod89] to hold under OCA, $C_k(X)$ is cosmic, which is equivalent to $C_k(X)$ being \aleph_0 . Then X is \aleph_0 , and hence $\mathcal{K}(X)$ is cosmic.

Now suppose $\mathcal{K}(X)$ is separable metrisable, then clearly $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc. Conversely, if $\mathcal{K}(X)$ is first countable then, by Proposition 18 of [GR], X has (CK), and, as above, $\mathcal{K}(X)$ is cosmic. Hence (Proposition 2.2.8) X is first countable and \aleph_0 . But first countable \aleph_0 spaces are separable metrisable (see [Gru84]). Finally, X is separable metrisable if and only if $\mathcal{K}(X)$ is separable metrisable. \square

Example 2.2.15 ($\mathfrak{b} = \omega_1$). There is an uncountable subset X of the Sorgenfrey Line such that $\mathcal{K}(X)^\omega$ is hereditarily ccc and first countable (and hence X has (CK)), but $\mathcal{K}(X)$ and X are not cosmic.

Construction. Using $\mathfrak{b} = \omega_1$, and observing that any discrete subspace is left separated, the space $X = A[\leq_{\text{lex}}]$ given by Todorčević in Theorem 3.0 of [Tod89] is a subspace of the Sorgenfrey Line (with left-facing topology) of size ω_1 such that X^n is hereditarily ccc for all n . It is clear that X , and therefore $\mathcal{K}(X)$, does not possess a countable network.

Observe that the compact subsets of X are homeomorphic to countable compact ordinals, and so a basic open neighbourhood of an element A of $\mathcal{K}(X)$ is composed of pairwise disjoint basic open intervals in the Sorgenfrey Line, whose right-hand end-

points are points of A . From this one can easily check that $\mathcal{K}(X)$ is first countable.

Suppose that \mathcal{A} were an uncountable discrete subspace of X . Then for every $A \in \mathcal{A}$ there is an open $\mathcal{U}^A = [U_1^A \dots U_{n_A}^A]$ such that $\mathcal{U}^A \cap \mathcal{A} = \{A\}$. We may assume that each \mathcal{U}^A is basic, and, by counting, that $n_A = n$ for all $A \in \mathcal{A}$. Let x_i^A be the first point of A in U_i^A , and y_i^A the corresponding right-hand end-point.

Then, setting $p(A) = (x_1^A, y_1^A, \dots, x_n^A, y_n^A)$, $P = \{p(A) : A \in \mathcal{A}\}$ is a subset of X^{2n} , each point with open neighbourhood $V^A = U_1^A \times U_1^A \times \dots \times U_n^A \times U_n^A$. If P is countable then distinct A, A' give the same points in X^{2n} , and hence $A \in \mathcal{U}^{A'}$. If P is uncountable, then since X^{2n} is hereditarily ccc, P cannot be discrete, so there exist distinct A, A' such that $p(A) \in V^{A'}$, and hence $A \in \mathcal{U}^{A'}$. This contradicts the discreteness of \mathcal{A} . Thus $\mathcal{K}(X)$ is hereditarily ccc. By a similar method, $\mathcal{K}(X)^n$ is hereditarily ccc for all $n \in \omega$. \square

Corollary 2.2.16. *The statement: ‘A space X is separable metrisable if and only if $\mathcal{K}(X)^\omega$ is first countable and hereditarily ccc’ is consistent and independent of Set Theory.*

2.2.7 Open Questions

The key remaining questions seem to be the following.

Question 7.1. Does $\mathcal{K}(X) \neq \mathcal{F}(X)$, and $\mathcal{K}(X)$ MN, imply that $\mathcal{K}(X)$ is stratifiable?

Question 7.2. Does $\mathcal{K}(X) \neq \mathcal{F}(X)$, and $\mathcal{K}(X)$ hereditarily normal, imply that $\mathcal{K}(X)$ is perfectly normal? What if X is non-trivial?

Problem 7.3. Determine those spaces X such that $\mathcal{K}(X)$ is stratifiable.

Remark. The results of subsections 2.2.3 and 2.2.4 were obtained independently by the first two and last two authors. Subsection 2.2.5 is joint work. The final subsection is due to the first two authors.

2.3 On perfect mappings between hyperspaces of compact subsets

We construct three perfect mappings between hyperspaces of non-empty compact subsets with finite topology. These are used to determine under what kind of conditions hyperspaces and mapping spaces become monotone normal, stratifiable and σ -spaces.

2.3.1 Introduction.

All spaces are assumed to be regular T_2 -spaces. For a space X , let $\mathcal{K}(X)$, $\mathcal{F}(X)$ be families of non-empty compact subsets, non-empty finite subsets of X , respectively. \mathbb{N} always denotes positive integers. We assume that $\mathcal{K}(X)$ has finite topology, that is, the base of which consists of all subsets of the form

$$\langle U_1, \dots, U_k \rangle = \left\{ K \in \mathcal{K}(X) \mid K \subset \bigcup_{i=1}^k U_i, K \cap U_i \neq \emptyset \text{ for each } i \right\},$$

where $\{U_1, \dots, U_k\}$ is a finite family of open subsets of X . We write $\langle U_i \mid i = 1, \dots, k \rangle$ or $\langle \mathcal{U} \rangle$ in place of $\langle U_1, \dots, U_k \rangle$, where $\mathcal{U} = \{U_1, \dots, U_k\}$. Let $C_K(X, Y)$ be a mapping space with compact-open topology, that is, it has the base for topology consisting of

$$W(K_1, \dots, K_n; U_1, \dots, U_n) = \{f \in C_K(X, Y) \mid f(K_i) \subset U_i \text{ for each } i\},$$

where $K_1, \dots, K_n \in \mathcal{K}(X)$, U_1, \dots, U_n are open in Y and $n \in \mathbb{N}$.

In this section, we establish three perfect mappings from hyperspaces to hyperspaces. These are applicable to the determination of monotone normality and stratifiability of hyperspaces. On the other hand, we establish the relation between hyperspaces and mapping spaces. As for undefined term, refer to [Gru84].

2.3.2 Operators on hyperspaces.

Theorem 2.3.1. *Let $\varphi : \mathcal{K}(X) \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ be defined by the following:*

$$\varphi((K, L)) = K \cup L, \quad K, L \in \mathcal{K}(X).$$

Then φ is a perfect mapping.

Proof. It is obvious that φ is onto. To see the continuity of φ , let $\hat{U} = \langle U_1, \dots, U_k \rangle$ be an open neighborhood of $\varphi((K, L)) = K \cup L$ in $\mathcal{K}(X)$. Let

$$\mathcal{U}(K) = \{U_i | U_i \cap K \neq \emptyset\},$$

$$\mathcal{U}(L) = \{U_i | U_i \cap L \neq \emptyset\}.$$

Then $\langle \mathcal{U}(K) \rangle, \langle \mathcal{U}(L) \rangle$ are open neighborhoods of K, L in $\mathcal{K}(X)$, respectively. It is easily checked that $\hat{W} = \langle \mathcal{U}(K) \rangle \times \langle \mathcal{U}(L) \rangle$ is an open neighborhood of (K, L) in $\mathcal{K}(X)^2$ such that $\varphi(\hat{W}) \subset \hat{U}$. Hence φ is continuous. Note that for each $K \in \mathcal{K}(X)$,

$$(1) \quad \varphi^{-1}(K) = \{(A_\lambda, B_\lambda) \in \mathcal{K}(X)^2 | A_\lambda \cup B_\lambda = K\}.$$

This means $\varphi^{-1}(K) \subset \mathcal{K}(X) \times \mathcal{K}(X)$, and therefore $\varphi^{-1}(K)$ is compact because $\mathcal{K}(K)$ is compact by [Mic51, Theorem 4.9]. To see the closedness of φ , let \hat{O} be an open neighborhood of $\varphi^{-1}(K)$ in $\mathcal{K}(X)^2$. As seen in (1), $\varphi^{-1}(K) = \{(A_\lambda, B_\lambda) | \lambda \in \Lambda\}$. For each λ , there exists an open neighborhood $\hat{U}(\lambda) = \langle \mathcal{U}(\lambda) \rangle \times \langle \mathcal{V}(\lambda) \rangle$ of (A_λ, B_λ) in $\mathcal{K}(X)^2$ such that $\hat{U}(\lambda) \subset \hat{O}$, where $\mathcal{U}(\lambda), \mathcal{V}(\lambda)$ are finite families of open subsets of X . Since $\varphi^{-1}(K)$ is compact, there exists a finite subset Λ_0 of Λ such that

$$(2) \quad \varphi^{-1}(K) \subset \bigcup \{\hat{U}(\lambda) | \lambda \in \Lambda_0\} \subset \hat{O}.$$

Let

$$\mathcal{W}' = \bigcup \{\mathcal{U}(\lambda) \cup \mathcal{V}(\lambda) | \lambda \in \Lambda_0\}.$$

Then \mathcal{W}' is a finite open cover of K in X . Let $\{P(\delta) | \delta \in \Delta\}$ be the partition of K by \mathcal{W}' , where $|\Delta| < \aleph_0$. (That is, for each δ ,

$$P(\delta) = K \cap \left(\bigcap \mathcal{W}(\delta) \setminus \bigcup (\mathcal{W}' \setminus \mathcal{W}(\delta)) \right)$$

for some $\mathcal{W}(\delta) \subset \mathcal{W}'$.) For each $\delta \in \Delta$, let

$$(3) \quad W(\delta) = \bigcup \{W \in \mathcal{W}' | P(\delta) \subset W\}.$$

Then $\mathcal{W} = \{W(\delta) | \delta \in \Delta\}$ is an open cover of K , which means that $\langle \mathcal{W} \rangle$ is an open neighborhood of K in $\mathcal{K}(X)$. To see $\varphi^{-1}(\langle \mathcal{W} \rangle) \subset \hat{O}$, let $(A, B) \in \varphi^{-1}(\langle \mathcal{W} \rangle)$. This means $A \cup B \in \langle \mathcal{W} \rangle$. Let

$$C = \bigcup \{P(\delta)^- | A \cap W(\delta) \neq \emptyset\},$$

$$D = \bigcup \{P(\delta)^- | B \cap W(\delta) \neq \emptyset\}.$$

Then $(C, D) \in \varphi^{-1}(K)$. By (2), there exists $\lambda_0 \in \Lambda_0$ such that $(C, D) \in \hat{U}(\lambda_0)$. Then we can show $A \in \langle \mathcal{U}(\lambda_0) \rangle$. Since $C \subset \bigcup \mathcal{U}(\lambda_0)$,

$$\bigcup \{P(\delta)^- | A \cap W(\delta) \neq \emptyset\} \subset \bigcup \mathcal{U}(\lambda_0).$$

This combined with the definition of $W(\delta)$ in (3) implies

$$A \subset \bigcup \{W(\delta) | A \cap W(\delta) \neq \emptyset\} \subset \bigcup \mathcal{U}(\lambda_0).$$

On the other hand, let $U \in \mathcal{U}(\lambda_0)$ be any element. Since $C \cap U \neq \emptyset$, there exists $\delta \in \Delta$ such that $P(\delta)^- \cap U \neq \emptyset$ and $A \cap W(\delta) \neq \emptyset$. $P(\delta)^- \cap U \neq \emptyset$ implies $U \cap P(\delta) \neq \emptyset$, which combined with (3) implies $W(\delta) \subset U$. Hence $A \cap U \neq \emptyset$. Thus we have $A \in \langle \mathcal{U}(\lambda_0) \rangle$. Similarly we can show $B \in \langle \mathcal{V}(\lambda_0) \rangle$. Hence we have $\varphi^{-1}(\langle \mathcal{W} \rangle) \subset \hat{O}$, proving that φ is a closed mapping. \square

Theorem 2.3.2. *Let $\varphi : \mathcal{K}(X \times Y) \rightarrow \mathcal{K}(X) \times \mathcal{K}(Y)$ be defined by the following:*

$$\varphi(K) = (\pi_1(K), \pi_2(K)), \quad K \in \mathcal{K}(X \times Y),$$

where $\pi_1 : X \times Y \rightarrow X$, $\pi_2 : X \times Y \rightarrow Y$ are natural projections. Then φ is a perfect mapping.

Proof. Obviously φ is onto. To see the continuity of φ , let $\varphi(K) = (L, M)$ and let $\langle \mathcal{U} \rangle \times \langle \mathcal{V} \rangle$ be any basic open neighborhood of (L, M) in $\mathcal{K}(X) \times \mathcal{K}(Y)$, where \mathcal{U}, \mathcal{V} are finite open covers of L, M , in X, Y , respectively. Let

$$\mathcal{W} = \{U \times V | (U \times V) \cap K \neq \emptyset, U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Then it is easy to see $\langle \mathcal{W} \rangle$ is an open neighborhood of K in $\mathcal{K}(X \times Y)$ such that $\varphi(\langle \mathcal{W} \rangle) \subset \langle \mathcal{U} \rangle \times \langle \mathcal{V} \rangle$. Let $(L, M) \in \mathcal{K}(X) \times \mathcal{K}(Y)$. Since $L \times M$ is compact, $\varphi^{-1}((L, M))$ is compact in $\mathcal{K}(X \times Y)$. We show that φ is closed. Let $(L, M) \in \mathcal{K}(X) \times \mathcal{K}(Y)$ and let \hat{O} be any open neighborhood of $\varphi^{-1}((L, M))$ in $\mathcal{K}(X \times Y)$. Note that

$$\varphi^{-1}((L, M)) = \{K_\alpha \in \mathcal{K}(X \times Y) \mid \pi_1(K_\alpha) = L, \pi_2(K_\alpha) = M\}.$$

Since $\varphi^{-1}((L, M))$ is compact, there exists a finite family $\{\hat{O}_\alpha \mid \alpha \in A_0\}$ of basic open subsets of $\mathcal{K}(X \times Y)$ such that

$$(1) \quad \varphi^{-1}((L, M)) \subset \bigcup \left\{ \hat{O}_\alpha \mid \alpha \in A_0 \right\} \subset \hat{O},$$

where for each $\alpha \in A_0$, $\hat{O}_\alpha = \langle U_{\alpha_i} \times V_{\alpha_i} \mid i = 1, \dots, t_\alpha \rangle$ and all $U_{\alpha_i}, V_{\alpha_i}$ are open in X, Y , respectively. Let

$$\mathcal{U}_\alpha = \{U_{\alpha_i} \mid i = 1, \dots, t_\alpha\}, \alpha \in A_0,$$

$$\mathcal{V}_\alpha = \{V_{\alpha_i} \mid i = 1, \dots, t_\alpha\}, \alpha \in A_0.$$

Then each $\mathcal{U}_\alpha, \mathcal{V}_\alpha$ are open cover of L, M , in X, Y , respectively. Let $\{P(\delta_i) \mid i = 1, \dots, m\}$ be the partition of L by $\mathcal{U}_0 = \bigcup \{\mathcal{U}_\alpha \mid \alpha \in A_0\}$, that is, for each i there exists $\mathcal{U}(\delta_i) \subset \mathcal{U}_0$ such that

$$\emptyset \neq P(\delta_i) = L \cap \left(\bigcap \mathcal{U}(\delta_i) \setminus \bigcup (\mathcal{U}_0 \setminus \mathcal{U}(\delta_i)) \right).$$

Let

$$U(\delta_i) = \bigcap \mathcal{U}(\delta_i), \quad i = 1, \dots, m.$$

$$L(\delta_i) = P(\delta_i)^-, \quad i = 1, \dots, m.$$

Then the following is easily checked:

$$(2) \quad L \in \langle U(\delta_i) \mid i = 1, \dots, m \rangle \text{ and}$$

for each $U \in \mathcal{U}_0$, $L(\delta_i) \cap U \neq \emptyset$ if and only if $U(\delta_i) \subset U$.

By the same method, we get a finite family $\{V(\mu_j)|j = 1, \dots, n\}$ of open subsets of Y and a finite cover $\{M(\mu_j)|j = 1, \dots, n\}$ of M by compact subsets of Y satisfying the following (3):

$$(3) \quad M \in \langle V(\mu_j)|j = 1, \dots, n \rangle \text{ and}$$

$$\text{for each } V \in \mathcal{V}_0 = \bigcup \{\mathcal{V}_\alpha | \alpha \in A_0\},$$

$$M(\mu_j) \cap V \neq \emptyset \text{ if and only if } V(\mu_j) \subset V.$$

Note that

$$\{L(\delta_i) \times M(\mu_j)|i = 1, \dots, m, j = 1, \dots, n\}$$

is a cover of $L \times M$ by compact subsets of $X \times Y$. We show the following inclusion:

$$\varphi^{-1}(\langle U(\delta_i)|i = 1, \dots, m \rangle \times \langle V(\mu_j)|j = 1, \dots, n \rangle) \subset \hat{O}.$$

To see it, let K' be any element of the left term above. If we let $\varphi(K') = (L', M')$, then

$$L' \in \langle U(\delta_i)|i = 1, \dots, m \rangle, \quad M' \in \langle M(\mu_j)|j = 1, \dots, n \rangle.$$

Let

$$N_0 = \{(i, j)|K' \cap (U(\delta_i) \times V(\mu_j)) \neq \emptyset\}.$$

Then $p_1(N_0) = \{1, \dots, m\}$, $p_2(N_0) = \{1, \dots, n\}$, where p_1, p_2 are projections such that $p_1((m', n')) = m'$, $p_2((m', n')) = n'$. It is easy to see

$$(4) \quad K' \in \langle U(\delta_i) \times V(\mu_j)|(i, j) \in N_0 \rangle.$$

Let

$$K_0 = \bigcup \{L(\delta_i) \times M(\mu_j)|(i, j) \in N_0\}.$$

Then $K_0 \in \mathcal{K}(X \times Y)$ and $\varphi(K_0) = (L, M)$. By (1), there exists $\alpha \in A_0$ such that $K_0 \in \hat{O}_\alpha$. Then we show $K' \in \hat{O}_\alpha$. To see it, let $(p, q) \in K'$. By (4), there exists

$(i, j) \in N_0$ such that $(p, q) \in U(\delta_i) \times V(\mu_j)$. On the other hand, since $K_0 \in \hat{O}_\alpha$, there exists $U \in \mathcal{U}_\alpha, V \in \mathcal{V}_\alpha$ with $U = U_{\alpha_k}, V = V_{\alpha_k}$ such that

$$(U \times V) \cap (L(\delta_i) \times M(\mu_j)) \neq \emptyset,$$

which combined with (2) and (3) implies

$$U(\delta_i) \times V(\mu_j) \subset U \times V.$$

Thus $(p, q) \in U \times V$. Let $U \in \mathcal{U}_\alpha, V \in \mathcal{V}_\alpha$ be any member such that $U = U_{\alpha_k}, V = V_{\alpha_k}$. Because $K_0 \in \hat{O}_\alpha$, there exists $(i, j) \in N_0$ such that

$$(U \times V) \cap (L(\delta_i) \times M(\mu_j)) \neq \emptyset.$$

Both $U \cap L(\delta_i) \neq \emptyset$ and $V \cap M(\mu_j) \neq \emptyset$ imply $U(\delta_i) \subset U$ and $V(\mu_j) \subset V$ because of (2) and (3), respectively. Since $K' \cap (U(\delta_i) \times V(\mu_j)) \neq \emptyset$ by (4), we have $K' \cap (U \times V) \neq \emptyset$. Hence $K' \in \hat{O}_\alpha$ is proved. This completes the proof that φ is closed. \square

Corollary 2.3.3. *Let $\varphi : \mathcal{K}(X^2) \rightarrow \mathcal{K}(X)$ be defined by the following:*

$$\varphi(K) = \pi_1(K) \cup \pi_2(K), \quad K \in \mathcal{K}(X^2),$$

where π_1, π_2 are projections onto X such that $\pi_1((x, y)) = x, \pi_2((x, y)) = y$. Then φ is a perfect mapping.

Proof. φ is considered to be the composition of both perfect mappings of the two theorems above. \square

In our earlier paper, we established following equivalence on monotone normality of hyperspaces $\mathcal{F}(X)$ of finite subsets of a space X : For a space X , the followings are equivalent: (i) X^2 is monotonically normal (for brevity, MN), (ii) X^n is MN for each $n \geq 2$, (iii) $\mathcal{F}(X)$ is MN, (iv) $\mathcal{F}(X)^n$ is MN for each $n \in \mathbb{N}$, [FGMS97, Theorem 3.1, Corollary 3.3]. Using this, we establish the following result of the similar type:

Corollary 2.3.4. *For a space X , the followings are equivalent:*

(i) $\mathcal{K}(X)$ is MN,

(ii) $\mathcal{K}(X^2)$ is MN,

(iii) $\mathcal{K}(X^n)$ is MN for each $n \in \mathbb{N}$ and

(iv) $\mathcal{K}(X)^n$ is MN for each $n \in \mathbb{N}$.

Proof. For the case of $\mathcal{K}(X) = \mathcal{F}(X)$, this is nothing but the above. Thus we suppose that there exists $K \in \mathcal{K}(X) \setminus \mathcal{F}(X)$ and we recall that for this case monotone normality and stratifiability of $\mathcal{K}(X)$ are equivalent [FGMS97, Theorem 4.2]. Therefore it suffices to show the equivalence of the following statements: (i)' $\mathcal{K}(X)$ is stratifiable, (ii)' $\mathcal{K}(X^2)$ is stratifiable, (iii)' $\mathcal{K}(X^n)$ is stratifiable for each $n \in \mathbb{N}$ and (iv)' $\mathcal{K}(X)^n$ is stratifiable for each $n \in \mathbb{N}$.

(iii)' \rightarrow (ii)' \rightarrow (i)' \rightarrow (iv)' is trivial. (iv)' \rightarrow (iii)': By Corollary 2.3.3, there exists a perfect mapping of $\mathcal{K}(X^2)$ onto $\mathcal{K}(X)$. Since X^2 is stratifiable, by [Miz96, Theorem 1(1)] $\mathcal{K}(X^2)$ has a G_δ -diagonal. By [Bor66, Theorem 8.4] $\mathcal{K}(X^2)$ is also stratifiable. By repeating the discussion, we can show that $\mathcal{K}(X^{2^n})$ is stratifiable for each $n \in \mathbb{N}$. Since $\mathcal{K}(X^m) \hookrightarrow \mathcal{K}(X^n)$ if $m < n$, $m, n \in \mathbb{N}$, for each $n \in \mathbb{N}$ $\mathcal{K}(X^n)$ is stratifiable. \square

Theorem 2.3.5. $C_K(X, Y) \hookrightarrow \mathcal{K}(X \times Y)$ if X is compact.

Proof. We define the embedding $G : C_K(X, Y) \rightarrow \mathcal{K}(X \times Y)$ as follows:

$$G(f) = \{(x, f(x)) \mid x \in X\}, \quad f \in C_K(X, Y).$$

Obviously G is one-to-one. To see the continuity, let $\hat{O} = \langle U_i \times V_i \mid i = 1, \dots, k \rangle$ be an open neighborhood of $G(f)$ in $\mathcal{K}(X \times Y)$. Take a finite cover $\{K_i \mid i = 1, \dots, k\}$ of X by compact subsets of X such that $\emptyset \neq K_i \subset U_i \cap f^{-1}(V_i)$ for each i . Then

$$W = W(K_1, \dots, K_k; V_1, \dots, V_k)$$

is an open neighborhood of f in $C_K(X, Y)$ such that $G(W) \subset \hat{O}$. To see the openness of G , let $W = W(K_1, \dots, K_k; V_1, \dots, V_k)$ be an open neighborhood of f

in $C_K(X, Y)$. Let $\mathcal{K} = \{K_i | i = 1, \dots, k\}$ and $K = \bigcup \mathcal{K}$. Let $\mathcal{P} = \{P(\delta) | \delta \in \Delta\}$ be the partition of K by \mathcal{K} , i.e., for each $\delta \in \Delta$, $P(\delta) = \bigcup \mathcal{K}(\delta) \setminus \bigcup (\mathcal{K} \setminus \mathcal{K}(\delta))$, where $\mathcal{K}(\delta)$ is the subfamily of \mathcal{K} . For each $\delta \in \Delta$, let $V(\delta) = \bigcap \{V_i | K_i \in \mathcal{K}(\delta)\}$ and let $U(\delta) = f^{-1}(V_\delta) \setminus \bigcup (\mathcal{K} \setminus \mathcal{K}(\delta))$. We define an open neighborhood \hat{O} of $G(f)$ in $G(C_K(X, Y))$ by the following: If $X = K$, then

$$\hat{O} = \langle U(\delta) \times V(\delta) | \delta \in \Delta \rangle \cap G(C_K(X, Y))$$

and otherwise

$$\hat{O} = \langle \{U(\delta) \times V(\delta) | \delta \in \Delta\} \cup \{(X \setminus K) \times Y\} \rangle.$$

Then it is easy to see that $\hat{O} \subset G(W)$. This completes the proof. \square

Corollary 2.3.6. *If X is a compact space, then $C_K(X, Y)$ is the subspace of the perfect pre-image of $\mathcal{K}(X) \times \mathcal{K}(Y)$.*

Proof. $\mathcal{K}(X \times Y)$ is the perfect pre-image of $\mathcal{K}(X) \times \mathcal{K}(Y)$ by Theorem 2.3.2. \square

Corollary 2.3.7. *If X is a compact metric space and $\mathcal{K}(Y)$ is stratifiable, then $C_K(X, Y)$ is stratifiable.*

Proof. Since $X \times Y$ is stratifiable, by [Miz96, Theorem 1(1)] $\mathcal{K}(X \times Y)$ has a G_δ -diagonal. Then by the above, $C_K(X, Y)$ is stratifiable. \square

Corollary 2.3.8. *If X is a compact metric space and $\mathcal{K}(Y)$ is a σ -space, then $C_K(X, Y)$ is a σ -space.*

Proof. Since $X \times Y$ is a σ -space, by [Gru84, Theorem 4.6] and [Miz96, Theorem 1(1)] $\mathcal{K}(X \times Y)$ has a G_δ -diagonal. Moreover $\mathcal{K}(X \times Y)$ is a Σ -space as the perfect pre-image of a σ -space [Gru84, pp. 450-451]. Hence $\mathcal{K}(X \times Y)$, and necessarily $C_K(X, Y)$, is a σ -space by [Gru84, Theorem 4.15]. \square

2.4 On stratifiability of mapping spaces

We give a mapping space $C(X, Y)$ that is not M_3 , where X is a compact metrizable space and Y has the weak topology with respect to compact metrizable spaces, and show that $C(X, Y)$ is M_3 if X is a compact metrizable space and $\mathcal{K}(Y)$ is M_3 .

2.4.1 Introduction.

All spaces are assumed to be regular T_1 -spaces. \mathbb{N} always denotes the all positive integers. The terms “locally finite”, “closure-preserving” are abbreviated to LF, CP, respectively. For a space X , let $\mathcal{K}(X)$ be the set of all non-empty compact subsets of X and $\mathcal{F}(X)$ the set of all non-empty finite subsets of X . We give $\mathcal{K}(X)$ finite topology in the sense of Michael [Mic51], the base for which consists of all subsets of the following type:

$$\begin{aligned} & \langle U_1, \dots, U_k \rangle \\ & = \left\{ K \in \mathcal{K}(X) \mid K \subset \bigcup \{U_i \mid i = 1, \dots, k\} \text{ and } K \cap U_i \neq \emptyset \text{ for each } i \right\}, \end{aligned}$$

where $\{U_1, \dots, U_k\} \subset \tau(X)$, the topology of X , and $k \in \mathbb{N}$. Let $C(X, Y)$ be the set of all continuous mappings of a space X into a space Y and it have compact open topology, the base for which consists of all subsets of the following type:

$$W(K_1, \dots, K_n; U_1, \dots, U_n) = \{f \in C(X, Y) \mid f(K_i) \subset U_i \text{ for } i = 1, \dots, n\},$$

where $\mathcal{K}_0 = \{K_1, \dots, K_n\} \in \mathcal{F}(\mathcal{K}(X))$ and $\mathcal{U}_0 = \{U_1, \dots, U_n\} \in \mathcal{F}(\tau(X))$. (For brevity, such subsets are written as $W(\mathcal{K}_0; \mathcal{U}_0)$.) We write the space with this topology by $C(X, Y)$ again. It is known that for regular spaces X, Y , $\mathcal{K}(X)$ and $C(X, Y)$ are regular, [Mic51, Theorem 4.9.10], [Eng88, Theorem 3.4.13].

Let $\{X_\alpha \mid \alpha \in A\}$ be a closed cover of a space X . We say that X has the weak topology with respect to $\{X_\alpha\}$, if any subset of X whose intersection with each X_α is closed in X_α is necessarily closed in X . We say that X is dominated by $\{X_\alpha\}$ is $B \subset X$ is closed whenever it has a closed intersection with each member of some

$\{X_\alpha | \alpha \in A'\}$ which covers B . If X is dominated by $\{X_\alpha\}$, then X has the weak topology with respect to $\{X_\alpha\}$, but the reverse is not true. In [Bor71, Question 2.3] Borges posed the following question: If X is dominated by a family of metrizable subspaces $\{X_\alpha\}$ and S is a compact space, then is $C(S, X)$ M_3 ? In this section, we construct a space X having the weak topology with respect to compact metrizable spaces such that $C(S, X)$ is not M_3 , where S is the compact convergent sequence. This is a negative partial answer to the Borges' question. As the positive result, we settle that if X is a compact metrizable space and $\mathcal{K}(Y)$ is M_3 , then $C(X, Y)$ is M_3 .

As for the definition of M_3 -spaces, refer to [Bor71].

2.4.2 Example.

Example 2.4.1. There exists a space X having the weak topology with respect to compact metrizable spaces such that $\mathcal{K}(X)$ is not M_3 .

Construction. Let us introduce the following notation:

$$\begin{aligned} N_1 &= \{1\}, N_2 = N_3 = \dots = \mathbb{N}, \\ T(n) &= \prod \{N_i | i = 1, \dots, n\}, n \in \mathbb{N}, \\ T(\omega) &= \prod \{N_i | i \in \mathbb{N}\}, \\ T &= \bigcup \{T(n) | n \in \mathbb{N}\} \cup T(\omega), \end{aligned}$$

$$\begin{aligned} &F(k_1 k_2 \dots k_{n-1}; k) \\ &= \left\{ p \in \left(\bigcup \{T(i) | i \geq n\} \right) \cup T(\omega) \mid \text{the first } k \text{ coordinates of } p \text{ is } k_1 k_2 \dots k_{n-1} k \right\}, \\ &\qquad\qquad\qquad (k_1 k_2 \dots k_{n-1}) \in T(n-1), k \in \mathbb{N}, \end{aligned}$$

$$T(\alpha) = \{\alpha\} \cup \{(n_1 n_2 \dots n_k) | k \in \mathbb{N}\}, \alpha = (n_1 n_2 \dots) \in T(\omega).$$

We topologize T by defining neighborhood bases as follows: All points of $\bigcup \{T(n) | n \in \mathbb{N}\}$ are isolated and for each $\alpha = (n_1 n_2 \dots) \in T(\omega)$, let the family

$$\{\{\alpha\} \cup \{(n_1 \dots n_k) | k \geq n\} | n \in \mathbb{N}\}$$

be the neighborhood base at α . Let $S = \{\omega\} \cup \mathbb{N}$ be a convergent sequence such that $n \rightarrow \omega$ as $n \rightarrow \infty$. Let X be the quotient space obtained from $T \times S$ by identifying $T(k) \times \{\omega\}$, $k \in \mathbb{N}$, and $T(\omega) \times \{\omega\}$ with $\omega(k)$, $k \in \mathbb{N}$, and $\omega(\omega)$, respectively. Let φ be the quotient mapping. Define subspace of X as follows:

$$\begin{aligned} X(k_1 k_2 \cdots k_{n-1}; k) &= \varphi(F(k_1 k_2 \cdots k_{n-1}; k) \times S), \quad (k_1 k_2 \cdots k_{n-1}) \in T(n-1), \quad k \in \mathbb{N}, \\ X[k_1 k_2 \cdots k_{n-1}] &= \bigcup \{X(k_1 k_2 \cdots k_{n-1}; k) \mid k \in \mathbb{N}\}, \quad (k_1 k_2 \cdots k_{n-1}) \in T(n-1), \\ X(\alpha) &= \varphi(T(\alpha) \times S), \quad \alpha \in T(\omega). \end{aligned}$$

Obviously each $X(\alpha)$ is a compact metrizable space and it is easily observed that X has the weak topology with respect to $\{X(\alpha) \mid \alpha \in T(\omega)\}$. We show that $\mathcal{K}(x)$ is not M_3 . For each $k \in \mathbb{N}$, let

$$L(k) = \{\omega(\omega)\} \cup \{\omega(t) \mid t \geq k\}.$$

Then $L(k) \in \mathcal{K}(X)$ for each k . Assume that $\mathcal{K}(X)$ is M_3 . Then $\{L(1)\}$ has a CP closed neighborhood base $\hat{\mathcal{B}}$ in $\mathcal{K}(X)$. For each $\hat{B} \in \hat{\mathcal{B}}$, let $B = \bigcup \{K \mid K \in \hat{B}\}$. Then B is a neighborhood of $L(1)$ in X . For each subfamily $\hat{\mathcal{B}}' \subset \hat{\mathcal{B}}$, we let $\mathcal{B}' = \{B \mid \hat{B} \in \hat{\mathcal{B}}'\}$. In the below, we show that the initial assumption leads to a contradiction. To this end, we settle the next claims:

Claim 1: There exists $p(1) = \varphi((1), s_1)$ with $s_1 \in \mathbb{N}$ and the subfamily $\hat{\mathcal{B}}(1)$ of $\hat{\mathcal{B}}$ satisfying the following (1), (2):

$$(1) \quad L(2) \cup \{p(1)\} \in \text{Int } \hat{B} \text{ for each } \hat{B} \in \hat{\mathcal{B}}(1).$$

(2) There exists $k_2 \in \mathbb{N}$ such that

$$\mathcal{B}(1) \mid X(1; k_2) \text{ is a neighborhood base of } L(2) \text{ in } X(1; k_2).$$

Proof of the claim: Assume the contrary. For each $s \in \mathbb{N}$, let

$$\hat{\mathcal{Q}}(s) = \{\hat{B} \in \hat{\mathcal{B}} \mid L(2) \cup \{\varphi((1), s)\} \in \text{Int } \hat{B}\}.$$

Then $\hat{\mathcal{B}} = \bigcup\{\hat{Q}(s)|s \in \mathbb{N}\}$. By the assumption, for each s there exists an open neighborhood $O(s)$ of $L(2)$ in $X(1; s)$ such that

$$(3) \quad B \cap X(1; s) \not\subset O(s) \text{ for each } B \in \mathcal{Q}(s).$$

Let $O = \bigcup\{O(s)|s \in \mathbb{N}\}$. Then O is an open neighborhood of $L(2)$ in $X[1]$. Since $\hat{\mathcal{B}}$ is a neighborhood base of $\{L(1)\}$ in $\mathcal{K}(X)$, there exists $\hat{B} \in \hat{\mathcal{B}}$ such that

$$L(2) \in \{K \cap X[1]|K \in \hat{B}\} \subset \langle O \rangle.$$

Take $s \in \mathbb{N}$ with $\hat{B} \in \hat{Q}(s)$. Then this is a contradiction to (3).

Claim 2: There exists $p(2) = \varphi((1k_2), s_2)$ with $s_1 < s_2 \in \mathbb{N}$ and $\hat{\mathcal{B}}(2) \subset \hat{\mathcal{B}}(1)$ satisfying the following (4), (5):

$$(4) \quad L(3) \cup \{p(1), p(2)\} \in \text{Int } \hat{B} \text{ for each } \hat{B} \in \hat{\mathcal{B}}(2).$$

(5) There exists $k_3 \in \mathbb{N}$ such that

$$\mathcal{B}(2)|X(1k_2; k_3) \text{ is a neighborhood base of } L(3) \text{ in } X(1k_2; k_3).$$

Proof of the claim: Assume the contrary. For each $s \in \mathbb{N}$, let

$$\hat{Q}(s) = \{\hat{B} \in \hat{\mathcal{B}}(1)|L(3) \cup \{p(1), \varphi((1k_2), s)\} \in \text{Int } \hat{B}\}.$$

Then by (1)

$$\hat{\mathcal{B}}(1) = \bigcup\{\hat{Q}(s)|s \in \mathbb{N}, s > s_1\}.$$

By assumption, for each $s > s_1$ there exists an open neighborhood $O(s)$ of $L(3)$ in $X(1k_2; s)$ such that

$$(6) \quad B \cap X(1k_2; s) \not\subset O(s) \text{ for each } B \in \mathcal{Q}(s).$$

For each $s \leq s_1$, let $O(s) = X(1k_2; s)$. Then $O = \bigcup\{O(s)|s \in \mathbb{N}\}$ is an open neighborhood of $L(3)$ in $X[1k_2]$. By (2), there exists $\hat{B} \in \hat{\mathcal{B}}(1)$ such that $B \cap$

$X(1; k_2) \subset O$. Take $s \in \mathbb{N}$ with $\hat{B} \in \hat{Q}(s)$ and $s_1 < s$. But this is a contradiction to (6).

Since $p(2) \neq \omega(2)$, by (2) there exists $\hat{B}(1) \in \hat{\mathcal{B}}(1)$ such that $p(2) \notin B(1)$. This implies

$$\langle X, \{p(2)\} \rangle \cap \hat{B}(1) = \emptyset.$$

On the other hand, by (1)

$$L(2) \cup \{p(1)\} \in \hat{B}(1).$$

By the same discussion as Claim 2, we can choose $p(3)$, $\hat{\mathcal{B}}(3)$ and k_4 satisfying the following (7), (8), (9):

$$(7) \quad p(3) = \varphi((1k_2k_3), s_3) \text{ with } s_1 < s_2 < s_3 \in \mathbb{N}.$$

$$(8) \quad \hat{\mathcal{B}}(3) \subset \hat{\mathcal{B}}(2) \text{ and for each } \hat{B} \in \hat{\mathcal{B}}(3)$$

$$L(4) \cup \{p(1), p(2), p(3)\} \in \text{Int } \hat{B}.$$

$$(9) \quad \mathcal{B}(3)|X(1k_2k_3; k_4) \text{ is a neighborhood base of } L(4) \text{ in } X(1k_2k_3; k_4).$$

Using (4) and (5), we can take $\hat{B}(2) \in \hat{\mathcal{B}}(2)$ such that:

$$L(3) \cup \{p(1), p(2)\} \in \hat{B}(2),$$

$$\langle X, \{p(3)\} \rangle \cap \hat{B}(2) = \emptyset.$$

Repeating this process, we can choose sequences $\{p(n)\}$, $\{k_n\}$, $\{\hat{B}(n)\}$ satisfying (10), (11), (12): For each n ,

$$(10) \quad p(n) = \varphi((1k_2 \cdots k_n), s_n) \text{ with } s_1 < s_2 < \cdots < s_n \in \mathbb{N},$$

$$(11) \quad L(n+1) \cup \{p(1), \dots, p(n)\} \in \hat{B}(n) \text{ and}$$

$$\langle X, \{p(n+1)\} \rangle \cap \hat{B}(n) = \emptyset,$$

$$(12) \quad \hat{B}(n) \in \hat{\mathcal{B}}.$$

Let

$$K = \{p(n) | n \in \mathbb{N}\} \cup \{\omega(\omega)\}.$$

Then by (10), $K \in \mathcal{K}(X)$. But by the latter equality of (11), $K \notin \hat{B}(n)$ for each n .

If we let

$$K(n) = L(n+1) \cup \{p(1), \dots, p(n)\}, \quad n \in \mathbb{N},$$

then by the former relation of (11) $K(n) \in \hat{B}(n)$ for each n . From the construction, we can observe that

$$\{K(n) | n \in \mathbb{N}\} \cup \{K\} \subset \mathcal{K}(X(\alpha))$$

for $\alpha = (1k_2k_3 \dots) \in T(\omega)$ and $K(n) \rightarrow K$ as $n \rightarrow \infty$. This is a contradiction because $\{\hat{B}(n)\}$ is CP in $\mathcal{K}(X)$ by (12). This completes the task. \square

Example 2.4.2. There exist a space X having the weak topology with respect to compact metrizable spaces and a convergent sequence S such that $C(S, X)$ is not M_3 .

Construction. We show that for some spaces X, S as in Example 2.4.1, $C(S, X)$ is not M_3 . So, we use the same notations as there. Assume that $C(S, X)$ is M_3 . Let C_0 be the subset of $C(S, X)$ consisting of all members $f \in C(S, X)$ satisfying either (1) or (2):

$$(1) \quad f(\omega) = \omega(\omega) \text{ and}$$

there exists $k \in \mathbb{N}$ such that $f(n) = \omega(n)$ for each $n \geq k$ and

$$f(n) \in \varphi(T(n) \times S \setminus \{\omega\}) \text{ for each } n < k.$$

$$(2) \quad f(\omega) = \omega(\omega) \text{ and } f(n) \in \varphi(T(n) \times (S \setminus \{\omega\})) \text{ for each } n \in \mathbb{N}.$$

Then the subspace C_0 is M_3 . Let f_0 be a fixed member of C_0 satisfying (1) such that $f_0(\omega) = \omega(\omega)$ and $f_0(n) = \omega(n)$ for each $n \in \mathbb{N}$. Then f_0 has a CP closed neighborhood base \mathcal{A} in C_0 . For each $A \in \mathcal{A}$, let

$$[A] = \{f(S) | f \in A\} \subset \mathcal{K}(X).$$

Then we show that $\{[A] | A \in \mathcal{A}\}$ is a CP closed neighborhood base of $L(1) = f_0(S)$ in $\mathcal{K}_0 = \{f(S) | f \in C_0\}$. To see that it is CP in \mathcal{K}_0 , let \mathcal{A}_0 be any subfamily of \mathcal{A} and suppose

$$f(S) \in \mathcal{K}_0 \setminus \bigcup \{[A] | A \in \mathcal{A}_0\}.$$

This implies $f \in C_0 \setminus \bigcup \{A | A \in \mathcal{A}_0\}$. Since $\bigcup \{A | A \in \mathcal{A}_0\}$ is closed in C_0 , there exists an open neighborhood O of f in C_0 such that

$$O \cap \left(\bigcup \{A | A \in \mathcal{A}_0\} \right) = \emptyset.$$

Taking $f \in C_0$ and the fact that

$$\varphi(T(n) \times (S \setminus \{\omega\})) \cap \varphi(T(m) \times (S \setminus \{\omega\})) = \emptyset, \quad n \neq m,$$

into account, we can choose the following basic open neighborhood of f in $C(S, X)$, contained in O :

$$W(\{1\}, \dots, \{k-1\}, \{\omega\} \cup \{\omega(k), \omega(k+1), \dots\}; \{f(1)\}, \dots, \{f(k-1)\}, W)$$

where $k \in \mathbb{N}$ and W is an open neighborhood of $\{\omega(\omega)\} \cup \{\omega(k), \omega(k+1), \dots\}$ in $\bigcup \{\varphi(T(n) \times (S \setminus \{\omega\})) | n \geq k\} \cup \varphi(T(\omega) \times S)$. These imply

$$\mathcal{K}_0 \cap \langle \{f(1)\}, \dots, \{f(k-1)\}, W \rangle \cap \left(\bigcup \{[A] | A \in \mathcal{A}_0\} \right) = \emptyset.$$

Therefore $\{[A] | A \in \mathcal{A}\}$ is CP in \mathcal{K}_0 . Similarly, the closedness of each $[A]$ follows. It is obvious that $\{[A] | A \in \mathcal{A}\}$ is a neighborhood base of $L(1)$ in \mathcal{K}_0 . Here, we note that the essential part of the discussion of Example 2.4.1 can apply to this case. Hence we can get a contradiction. \square

2.4.3 Stratifiability of $C(X, Y)$ via $\mathcal{K}(X)$.

In spite of the result in Corollary 2.3.7, we state the following with the direct proof:

Theorem 2.4.3. *If X is a compact metrizable space and $\mathcal{K}(Y)$ is M_3 , then $C(X, Y)$ is M_3 .*

Proof. Since $Y \hookrightarrow \mathcal{K}(Y)$, Y is M_3 , and consequently submetrizable. Therefore there exists a sequence $\{\mathcal{U}(n) | n \in \mathbb{N}\}$ of LF open covers of Y satisfying the following:

- (1) For each n , $\mathcal{U}(n+1) < \mathcal{U}(n)$ and if $y \neq y'$, $y, y' \in Y$, there exists $n \in \mathbb{N}$ and neighborhoods $V(y), V(y')$ of y, y' in Y , respectively, such that

$$\text{St}(V(y), \mathcal{U}(n)) \cap V(y') = \emptyset.$$

(This condition is nothing but that Y has a regular G_δ -diagonal in the sense of Zenor [Zen72].) Since X is compact, by [Eng88, 3.12.27(j)] $C(X, Y)$ is embedded into $\mathcal{K}(X \times Y)$ by the embedding $G : C(X, Y) \rightarrow \mathcal{K}(X \times Y)$ defined by

$$G(f) = \{(x, f(x)) | x \in X\}, \quad f \in C(X, Y).$$

So, it suffices to show that $G(C(X, Y))$ is M_3 , that is, it has a σ -CP quasi-base. Since X is compact metrizable, there exists a countable base \mathcal{O} for X closed under finite unions. Let \mathcal{K} be the family of closures of members of \mathcal{O} . For each n , let $\Delta(n)$ be the totality of pairs $\delta = (\mathcal{K}(\delta), \mathcal{U}(\delta))$, where $\mathcal{K}(\delta) \in \mathcal{F}(\mathcal{K})$ is a cover of X and $\mathcal{U}(\delta) \in \mathcal{F}(\mathcal{U}(n))$, the members of which are in a one-to-one correspondence. For each $\delta = (\mathcal{K}(\delta), \mathcal{U}(\delta)) \in \Delta(n)$ with $\mathcal{K}(\delta) = \{K_1, \dots, K_s\}$ and $\mathcal{U}(\delta) = \{U_1, \dots, U_s\}$, let

$$V(\delta) = \bigcap_{i=1}^s (X \times Y \setminus (K_i \times (Y \setminus U_i))).$$

Then it is easy to see that $\mathcal{V}(n) = \{V(\delta) | \delta \in \Delta(n)\}$ is an open cover of $G(C(X, Y))$ in $\mathcal{K}(X \times Y)$. We define a subspace \mathcal{K}_0 as follows:

$$\mathcal{K}_0 = \{L \in \mathcal{K}(X \times Y) | \pi_X(L) = X\},$$

where π_X is the projection onto X . Then it is easily observed that \mathcal{K}_0 is closed in $\mathcal{K}(X \times Y)$. We settle the next two claims:

Claim 1: $\mathcal{V} = \bigcup\{\mathcal{V}(n)|n \in \mathbb{N}\}$ has the following property:

- (2) If $f \in C(X, Y)$ and $L \in \mathcal{K}_0$ such that $G(f) \neq L$, then there exists $V \in \mathcal{V}$ such that $G(f) \in V$ and $\text{Cl}_{\mathcal{K}_0}(V \cap \mathcal{K}_0) \subset \mathcal{K}_0 \setminus \{L\}$.

Proof of the claim: There exists $(x, y) \in L \setminus G(f)$. This implies $y \neq f(x)$. Therefore by (1) there exist $n \in \mathbb{N}$ and open neighborhoods $V(y)$, $V(f(x))$ of y , $f(x)$ in Y , respectively, such that

$$\text{St}(V(f(x)), \mathcal{U}(n)) \cap V(y) = \emptyset.$$

Take a cover $\{K_1, K_2\}$ of X such that $K_1, K_2 \in \mathcal{K}$, $x \in \text{Int } K_1 \subset K_1 \subset f^{-1}(V(f(x)))$, $x \in X \setminus K_2$ and $X \setminus \text{Int } K_1 \subset K_2$. Then it is easy to see that $\langle X \times Y, (X \setminus K_2) \times V(y) \rangle$ is an open neighborhood of L in $\mathcal{K}(X \times Y)$ such that

$$\langle X \times Y, (X \setminus K_2) \times V(y) \rangle \cap \text{St}(G(f), \mathcal{V}(n)) = \emptyset.$$

From this relation, any member V of $\mathcal{V}(n)$ containing $G(f)$ satisfies (2).

Claim 2: $\mathcal{V}|G(C(X, Y))$ is a σ -LF family of open subsets of $G(C(X, Y))$.

Proof of the claim: Note that

$$G(C(X, Y)) \cap V(\delta) = G(W(\mathcal{K}(\delta); \mathcal{U}(\delta)))$$

for each $\delta \in \Delta(n)$, $n \in \mathbb{N}$. So, it suffices to show that for each cover $\mathcal{K}^* = \{K_1, \dots, K_s\} \in \mathcal{F}(\mathcal{K})$ and $n \in \mathbb{N}$,

$$\mathcal{V}_0 = \{W(\mathcal{K}^*; \mathcal{U}(\delta)) | \delta \in \Delta(n), \mathcal{U}(\delta) = \{U_1, \dots, U_s\} \in \mathcal{F}(\mathcal{U}(n))\}$$

is LF in $C(X, Y)$. To this end, let $f_0 \in C(X, Y)$. Since X is compact, there exists a finite open cover $\{O(i) | i = 1, \dots, k\}$ of $f_0(X)$ in Y such that for each i

$$\mathcal{U}(i) = \{U \in \mathcal{U}(n) | U \cap O(i) \neq \emptyset\}$$

is finite. Then $\mathcal{U}_0 = \bigcup\{\mathcal{U}(i) \mid i = 1, \dots, k\}$ is also finite. Take $C_1, \dots, C_k \in \mathcal{K}(X)$ such that

$$f_0 \in W(C_1, \dots, C_k; O(1), \dots, O(k)) = \hat{W}.$$

To see that \hat{W} intersects only finitely many members of \mathcal{V}_0 , suppose $\hat{W} \cap W(\mathcal{K}^*; \mathcal{U}(\delta)) \neq \emptyset$, where $W(\mathcal{K}^*; \mathcal{U}(\delta)) \in \mathcal{V}_0$. Take $g \in \hat{W} \cap W(\mathcal{K}^*; \mathcal{U}(\delta))$. Then for each $K_j \in \mathcal{K}^*$, $K_j \cap C_i \neq \emptyset$ for some i , which means $U_j \cap O(i) \neq \emptyset$. Hence $\mathcal{U}(\delta) \subset \mathcal{U}_0$.

Finally, we show that $G(C(X, Y))$ is M_3 . By [SM98, Theorem 2.2] there exists a perfect mapping φ of $\mathcal{K}(X \times Y)$ onto $\mathcal{K}(Y)$. Let $\varphi_0 = \varphi|_{\mathcal{K}_0}$. Then φ_0 is also perfect because \mathcal{K}_0 is closed. By the assumption, there exists a σ -CP closed quasi-base \mathcal{B} for $\varphi_0(\mathcal{K}_0)$. Let \mathcal{W} be the family of all finite intersections of members of \mathcal{V} . By Claim 2, $\mathcal{W}|_{G(C(X, Y))}$ is also σ -LF in $G(C(X, Y))$. We define

$$\mathcal{P} = (\varphi_0^{-1}(\mathcal{B}) \cup (\varphi_0^{-1}(\mathcal{B}) \wedge \mathcal{W}))|_{G(C(X, Y))}.$$

Then it is easily checked that \mathcal{P} is a σ -CP family in $G(C(X, Y))$. To see that \mathcal{P} is a quasi-base for $G(C(X, Y))$, let $f \in C(X, Y)$ and \hat{O} an open neighborhood of $G(f)$ in \mathcal{K}_0 . As the first case, suppose $\varphi_0^{-1}(\varphi_0(G(f))) \subset \hat{O}$. Then $\varphi_0^*(\hat{O}) = \varphi_0(\mathcal{K}_0 \setminus \varphi_0(\mathcal{K}_0 \setminus \hat{O}))$ is an open neighborhood of $\varphi_0(G(f))$ in $\varphi_0(\mathcal{K}_0)$. There exists $B \in \mathcal{B}$ such that

$$\varphi_0(G(f)) \in \text{Int } B \subset B \subset \varphi_0^*(\hat{O}).$$

This implies that

$$G(f) \in \text{Int } P \subset P \subset \hat{O} \cap G(C(X, Y)),$$

where $P = \varphi_0^{-1}(B) \cap G(C(X, Y)) \in \mathcal{P}$. As the second case, suppose $\varphi_0^{-1}(\varphi_0(G(f))) \not\subset \hat{O}$. By Claim 1, there exists $W \in \mathcal{W}$ with $G(f) \in W$ and an open neighborhood \hat{G} of $\varphi_0^{-1}(\varphi_0(G(f))) \setminus \hat{O}$ in \mathcal{K}_0 such that $W \cap \hat{G} = \emptyset$. By the same discussion as the first case, we can take $B \in \mathcal{B}$ such that

$$G(f) \in \text{Int } \varphi_0^{-1}(B) \subset \varphi_0^{-1}(B) \subset \hat{O} \cup \hat{G}.$$

Then $P = \varphi_0^{-1}(B) \cap W \cap G(C(X, Y)) \in \mathcal{P}$ satisfies

$$G(f) \in \text{Int } P \subset P \subset \hat{O} \cap G(C(X, Y)).$$

Thus we have shown that \mathcal{P} is a σ -CP quasi-base for $G(C(X, Y))$, completing our task. □

2.5 On the embedding and developability of mapping spaces with compact open topology

We investigate the relation of mapping spaces with compact open topology and hyperspaces of compact subsets with finite topology. Using one of the results, we show the Moore spaces with a regular G_δ -diagonal are hereditary to this mapping spaces.

2.5.1 Introduction.

All spaces are assumed to be regular T_2 . For a space X , we denote by $\tau(X)$ the topology of X . Throughout this section, letter \mathbb{N} means the set of all positive integers. For families \mathcal{U}, \mathcal{V} of subsets of X , $\mathcal{U} < \mathcal{V}$ means that for each $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $U \subset V$. Let $\mathcal{K}(X)$ be the set of all non-empty compact subsets and for the topology of $\mathcal{K}(X)$, we use here the finite topology, which has the base consisting of all subsets of the form

$$\begin{aligned} &\langle U_1, \dots, U_k \rangle \\ &= \left\{ K \in \mathcal{K}(X) \mid K \subset \bigcup \{U_i \mid i = 1, \dots, k\} \text{ and } K \cap U_i \neq \emptyset \text{ for each } i \right\}. \end{aligned}$$

where $U_1, \dots, U_k \in \tau(X)$, $k \in \mathbb{N}$. For brevity, we use frequently the notation $\langle \mathcal{U} \rangle$ or $\langle U_i \mid i = 1, \dots, k \rangle$ in place of $\langle U_1, \dots, U_k \rangle$, where $\mathcal{U} = \{U_1, \dots, U_k\}$. As known already, if X is regular T_2 , then so is $\mathcal{K}(X)$. As for the fundamental properties of $\mathcal{K}(X)$, refer to [Mic51]. For spaces X, Y , let $C(X, Y)$ be the set of all continuous mappings of X into Y . As the topology of $C(X, Y)$, we accept the compact open topology, which has the base consisting of subsets of the form

$$W(K_1, \dots, K_n; O_1, \dots, O_n) = \{f \in C(X, Y) \mid f(K_i) \subset O_i \text{ for each } i\},$$

where $K_i \in \mathcal{K}(X)$ and $O_i \in \tau(Y)$ for each i . This space is written as $C_k(X, Y)$.

In the first part, we investigate the relation between mapping spaces and hyperspaces. We show that for a space X , $C_k(X, Y)$ is embedded into the product spaces

of hyperspaces. This embedding is shown to have some additional properties for special spaces X . The main result is used in the second part.

Next, we consider the classical problem on heredity of topological properties: Let \mathcal{P} be a class of spaces and let X be a compact or hemicompact space. If $Y \in \mathcal{P}$, then does $C_k(X, Y) \in \mathcal{P}$? Until now, we have some results for \mathcal{P} of metric spaces [Are46], \aleph_0 -spaces [Mic51] or paracompact \aleph -spaces [O'M71] etc. But we do not know whether Moore spaces are hereditary to $C_k(X, Y)$ when X is a compact space. Here, we show that this is the case for Moore spaces with a regular G_δ -diagonal.

As for undefined terms such as G_δ -diagonals, $w\Delta$ -spaces, etc, refer to [Gru84].

2.5.2 The embedding of $C_k(X, Y)$ into hyperspaces.

Theorem 2.5.1. *Let \mathcal{K} be a compact cover of a space X such that $\mathcal{K}(X) < \mathcal{K}$. Then for a space Y ,*

$$C_k(X, Y) \hookrightarrow \prod \{\mathcal{K}(K \times Y) \mid K \in \mathcal{K}\}.$$

Proof. Define $G : C(P, Q) \rightarrow 2^{P \times Q}$ as $G(f) = \{\langle x, f(x) \rangle \mid x \in P\}$, $f \in C(P, Q)$. We define the embedding $\varphi : C_k(X, Y) \rightarrow \prod \{\mathcal{K}(K \times Y) \mid K \in \mathcal{K}\}$ as follows:

$$\varphi(f) = \prod \{G(f|K) \mid K \in \mathcal{K}\}, \quad f \in C(X, Y).$$

Since \mathcal{K} covers X , it is easy to see that φ is one-to-one. To see the continuity of φ , it suffices to show that for each $K \in \mathcal{K}$, $G_K = \pi_K \cdot \varphi : C_k(X, Y) \rightarrow \mathcal{K}(K \times Y)$ is continuous, where $\pi_K : \prod \{\mathcal{K}(K \times Y) \mid K \in \mathcal{K}\} \rightarrow \mathcal{K}(K \times Y)$ is the projection.

Let

$$\hat{O} = \langle U_1 \times V_1, \dots, U_n \times V_n \rangle$$

be an open neighborhood of $G_K(f)$ in $\mathcal{K}(K \times Y)$, where $U_i \in \tau(K)$ and $V_i \in \tau(Y)$ for each i . Take a finite compact cover $\{K_i \mid i = 1, \dots, n\}$ of K such that $\emptyset \neq K_i \subset U_i \cap f^{-1}(V_i)$ for each i . Then

$$\hat{W} = W(K_1, \dots, K_n; V_1, \dots, V_n)$$

is an open neighborhood of f in $C_k(X, Y)$ such that $G_K(\hat{W}) \subset \hat{O}$.

To see the openness of φ , let

$$\hat{W} = W(K_1, \dots, K_n; V_1, \dots, V_n)$$

be an open neighborhood of f in $C_k(X, Y)$, where $K_i \in \mathcal{K}(X)$ and $V_i \in \tau(Y)$ for each i .

Let $\mathcal{P} = \{P(\delta) | \delta \in \Delta\}$ be the partition of $K = \bigcup\{K_i | i = 1, \dots, n\}$, i.e., for each $\delta \in \Delta$,

$$\emptyset \neq P(\delta) = \bigcap \mathcal{K}(\delta) \setminus \bigcup (\{K_i\} \setminus \mathcal{K}(\delta)),$$

where $\mathcal{K}(\delta)$ is a subfamily of $\{K_i\}$. Take $L \in \mathcal{K}$ such that $K \subset L$. For each δ , let

$$\begin{aligned} V(\delta) &= \bigcap \{V_i | K_i \in \mathcal{K}(\delta)\}, \\ U(\delta) &= f^{-1}(V(\delta)) \setminus \left(\bigcup (\{K_i\} \setminus \mathcal{K}(\delta)) \right). \end{aligned}$$

We define an open neighborhood \hat{O} of $\varphi(f)$ in $\varphi(C_k(X, Y))$ by the following:

$$\hat{O} = \begin{cases} \varphi(C_k(X, Y)) \cap \pi_L^{-1}(\{U(\delta) \times V(\delta) | \delta \in \Delta\}) & \text{if } L \setminus K = \emptyset, \\ \varphi(C_k(X, Y)) \cap \pi_L^{-1}(\{U(\delta) \times V(\delta) | \delta \in \Delta\} \cup \{(L \setminus K) \times Y\}) & \text{if } L \setminus K \neq \emptyset. \end{cases}$$

Then it is easy to see that $\hat{O} \subset \varphi(\hat{W})$. This completes the proof. \square

We remark that by virtue of the above theorem, when X is compact, $C_k(X, Y)$ is embedded into $\mathcal{K}(X \times Y)$ with the embedding $G : C_k(X, Y) \rightarrow \mathcal{K}(X \times Y)$ such that for each $f \in C_k(X, Y)$, $G(f)$ is the graph of f .

A space X is called *hemicompact* [Are46] if there exists a countable compact cover \mathcal{K} of X such that $\mathcal{K}(X) < \mathcal{K}$.

Corollary 2.5.2. *If X is a hemicompact space, then for a space Y $C_k(X, Y) \hookrightarrow \prod\{\mathcal{K}(K_i \times Y) | i \in \mathbb{N}\}$, where $\{K_i | i \in \mathbb{N}\} \subset \mathcal{K}(X)$.*

In the embedding theorem, we do not have any information on what kind of a subset of $\mathcal{K}(X \times Y)$ $\varphi(C_k(X, Y))$ is. The next two theorems give it for compact spaces X .

Theorem 2.5.3. *If X is a compact metrizable space and Y has a G_δ -diagonal, then $C_k(X, Y)$ is homeomorphic to a G_δ -set of $\mathcal{K}(X \times Y)$.*

Proof. Let \mathcal{B} be a countable quasi-base for X such that $\mathcal{B} \subset \mathcal{K}(X)$ and $\text{Int } B \neq \emptyset$ for each $B \in \mathcal{B}$. Assume that \mathcal{B} is closed under finite unions. Let $\{\mathcal{U}_n | n \in \mathbb{N}\}$ be a G_δ -diagonal sequence for Y . Let $n \in \mathbb{N}$. For each $f \in C(X, Y)$, take a finite subfamily $\mathcal{U}(f) = \{U_1, \dots, U_k\}$ of \mathcal{U}_n such that $f(X) \in \langle \mathcal{U}(f) \rangle$. We show that there exists a cover $\mathcal{B}(\delta) = \{B_1, \dots, B_k\}$ of X satisfying: $\mathcal{B}(\delta) \subset \mathcal{B}$ and

$$(1) \quad G(f) \in \bigcap \{ \langle X \times Y \setminus (B_i \times (Y \setminus U_i)) \rangle | i = 1, \dots, k \}.$$

Since f is continuous, there exists a finite compact cover $\{C_i | i = 1, \dots, k\}$ of X such that $C_i \subset f^{-1}(U_i)$ for each i . Since C_i is compact, there exists $B_i \in \mathcal{B}$ such that $C_i \subset B_i \subset f^{-1}(U_i)$. Thus we have $\mathcal{B}(\delta)$ satisfying (1). We write the right of (1) by $W(f, n)$ and set

$$W(n) = \bigcup \{ W(f, n) | f \in C(X, Y) \}.$$

Then each $W(n)$ is open in $\mathcal{K}(X \times Y)$ and

$$(2) \quad G(C_k(X, Y)) \subset \bigcap \{ W(n) | n \in \mathbb{N} \} \cap \bigcap \{ \langle X \times Y, \text{Int } B \times Y \rangle | B \in \mathcal{B} \}.$$

We show the converse of (2). Let L be any member of the right of (2). We note that $(\{x\} \times Y) \cap L \neq \emptyset$ for each $x \in X$, because $L \in \langle X \times Y, \text{Int } B \times Y \rangle$ for each B . For the first case, assume that $|L \cap (\{x\} \times Y)| \geq 2$ for some $x \in X$. Take $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in L$, where $y_1 \neq y_2$. Then there exists $n \in \mathbb{N}$ such that $y_2 \notin \text{St}(y_1, \mathcal{U}_n)$. For this n , $L \in W(n)$ means $L \in W(f, n)$ for some $f \in C(X, Y)$, which has the form by the definition as follows:

$$W(f, n) = \bigcap \{ \langle X \times Y \setminus (B_i \times (Y \setminus U_i)) \rangle | i = 1, \dots, k \},$$

where $\{U_1, \dots, U_k\} \subset \mathcal{U}_n$, $\{B_1, \dots, B_k\} \subset \mathcal{B}$. For some i , $x \in B_i$. Thus $L \in W(f, n)$ means $L \cap (B_i \times (Y \setminus U_i)) = \emptyset$, but this is a contradiction.

Hence we conclude that L has the form

$$L = \{\langle x, g(x) \rangle \mid x \in X\}$$

for some correspondence $g : X \rightarrow Y$. To see $g \in C(X, Y)$, assume the contrary. Since X is metrizable, there exists a sequence $\{x_n \mid n \in \mathbb{N}\}$ of points of X such that $x_n \rightarrow x$ but $g(x_n) \not\rightarrow g(x)$ as $n \rightarrow \infty$. Note that $\pi(L)$ is metrizable by [Gru84, Theorem 2.13], where $\pi : X \times Y \rightarrow Y$ is the projection. So, there exists a subsequence $\{x_{n(k)} \mid k \in \mathbb{N}\}$ such that $g(x_{n(k)}) \rightarrow y \neq g(x)$ as $k \rightarrow \infty$. Take n such that $g(x) \notin \text{St}(y, \mathcal{U}_n)$. But, on the other hand, since $L \in W(f, n)$ for some f , this is a contradiction. Thus we have

$$G(C_k(X, Y)) = \bigcap W(n) \cap \bigcap \langle X \times Y, \text{Int } B \times Y \rangle. \quad \square$$

We state the definition of being equicontinuous of $C_k(X, Y)$.

Let $\mathcal{F} \subset C(X, Y)$, where X is a space and Y is a uniform space with the uniformity $\mu = \{\mathcal{U}_\alpha \mid \alpha \in A\}$. If for each $\alpha \in A$ and each $p \in X$, there exists a neighborhood $N(p)$ of p such that

$$f(N(p)) \subset \text{St}(f(p), \mathcal{U}_\alpha)$$

for each $f \in \mathcal{F}$, then \mathcal{F} is called *equicontinuous*, [Nag85, p. 282].

Theorem 2.5.4. *Let X be a compact space and Y be a uniform space. If $C(X, Y)$ is equicontinuous, then $G(C_k(X, Y))$ is a closed subspace of $\mathcal{K}(X \times Y)$, i.e., $C_k(X, Y)$ is embedded into a closed subspace of $\mathcal{K}(X \times Y)$.*

Proof. We show that $G(C_k(X, Y))$ is closed in $\mathcal{K}(X \times Y)$. Take $L \in \mathcal{K}(X \times Y) \setminus G(C_k(X, Y))$. Suppose $|L \cap (\{x\} \times Y)| \geq 2$ for some $x \in X$. Take $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in L$ with $y_1 \neq y_2$. Let $\mu = \{\mathcal{U}_\alpha \mid \alpha \in A\}$ be the uniformity of Y compatible with Y . Then there exists $\mathcal{U} \in \mu$ such that $y_1 \notin \text{St}(y_2, \mathcal{U})$ and let \mathcal{V} be an open cover of Y such that $\mathcal{V} \in \mu$ and $\mathcal{V}^{**} < \mathcal{U}$. For this \mathcal{V} , there exists an open neighborhood $N(x)$ of x in X such that $f(N(x)) \subset \text{St}(f(x), \mathcal{V})$ for each $f \in C(X, Y)$. Take

$V_1, V_2 \in \mathcal{V}$ such that $y_1 \in V_1, y_2 \in V_2$. Then it is easy to see

$$\hat{O} = \langle N(x) \times V_1, N(x) \times V_2, X \times Y \rangle$$

is an open neighborhood of L in $\mathcal{K}(X \times Y)$ such that $\hat{O} \cap G(C_k(X, Y)) = \emptyset$. Next, suppose $|L \cap (\{x\} \times Y)| \leq 1$ for each $x \in X$. If there exists $x \in X$ such that $L \cap (\{x\} \times Y) = \emptyset$. Then it is easy to see that $\langle (X \setminus \{x\}) \times Y \rangle$ is an open neighborhood of L in $\mathcal{K}(X \times Y)$ missing $G(C_k(X, Y))$. For the last case, we suppose $L = \{\langle x, b(x) \rangle | x \in X\}$. Since $L \notin G(C_k(X, Y))$, the correspondence b is not continuous. This means that for some $A \subset X$ there exists $y \in b(\overline{A}) \setminus \overline{b(A)}$. Let $y = b(x)$ with $x \in \overline{A}$. There exists $\mathcal{U} \in \mu$ and an open cover $\mathcal{V} \in \mu$ such that $\text{St}(y, \mathcal{U}) \cap b(A) = \emptyset$ and $\mathcal{V}^{**} < \mathcal{U}$. Since $C_k(X, Y)$ is equicontinuous, there exists an open neighborhood $N(x)$ of x in X such that $f(N(x)) \subset \text{St}(f(x), \mathcal{V})$. Take $V, V' \in \mathcal{V}$ such that $y \in V, b(x_0) \in V'$, where $x_0 \in N(x) \cap A$. Then it is easy to see that $\langle N(x) \times V, N(x) \times V', X \times Y \rangle$ is an open neighborhood of L in $\mathcal{K}(X \times Y)$ missing $G(C_k(X, Y))$. This completes the proof. \square

Lemma 2.5.5. *Let \mathcal{K} be a compact cover of a space X such that $\mathcal{K}(X) < \mathcal{K}$. Then*

$$C_k(X, Y) \hookrightarrow \prod \{C_k(K, Y) | K \in \mathcal{K}\}.$$

This follows by the same way as in the proof of (g)→(a) in [MN86, Theorem 3.2].

2.5.3 Mapping spaces and Moore spaces.

Let us recall the definition of a regular G_δ -diagonal: A space X has a *regular G_δ -diagonal* if the diagonal set of $X \times X$ is a regular G_δ -set, and equivalently, if there exists a sequence $\{\mathcal{U}(n) | n \in \mathbb{N}\}$ of open covers of X such that if $x \neq y, x, y \in X$, then there exists $n \in \mathbb{N}$ and open neighborhoods O, O' of x, y in X , respectively, such that $\text{St}(O, \mathcal{U}(n)) \cap O' = \emptyset$, [Zen72, Theorem 1]. In this characterization, we can assume $\mathcal{U}(n+1) < \mathcal{U}(n), n \in \mathbb{N}$, and this is assumed in the sequel without any specification.

Theorem 2.5.6. *If X is a compact space and Y has a regular G_δ -diagonal (G_δ -diagonal, G_δ^* -diagonal), then $C_k(X, Y)$ has a regular G_δ -diagonal (G_δ -diagonal, G_δ^* -diagonal, respectively).*

Proof. We show the case of a regular G_δ -diagonal and the others are the same. By the characterization, there exists a sequence $\{\mathcal{U}(n) | n \in \mathbb{N}\}$ of open covers of Y such that if $y \neq y'$, $y, y' \in Y$, then there exists $n \in \mathbb{N}$ and open neighborhoods O, O' of y, y' in Y , respectively, such that $\text{St}(O, \mathcal{U}(n)) \cap O' = \emptyset$. We construct a sequence $\{\mathcal{W}(n) | n \in \mathbb{N}\}$ of open covers of $C_k(X, Y)$ by the following method (*) which is used later frequently:

(*) Let $n \in \mathbb{N}$ and $\{\delta = (\mathcal{K}(\delta), \mathcal{U}(\delta)) | \delta \in \Delta(n)\}$ be the totality of pairs of subfamilies $\mathcal{K}(\delta), \mathcal{U}(\delta)$ of $\mathcal{K}(X), \mathcal{U}(n)$, respectively, such that $\mathcal{K}(\delta) = \{K_1, \dots, K_t\}$ is a finite cover of X and $\mathcal{U}(\delta) = \{U_1, \dots, U_t\}$. For each $\delta \in \Delta(n)$, let

$$W(\delta) = W(K_1, \dots, K_t; U_1, \dots, U_t)$$

and $\mathcal{W}(n) = \{W(\delta) | \delta \in \Delta(n)\}$.

Since X is compact, for each $f \in C_k(X, Y)$ and $n \in \mathbb{N}$, we can easily find $\delta \in \Delta(n)$ such that $f \in W(\delta)$. Thus each $\mathcal{W}(n)$ is an open cover of $C_k(X, Y)$. Suppose $f \neq g$, $f, g \in C(X, Y)$. Then $f(x_0) \neq g(x_0)$ for some x_0 . By the property of $\{\mathcal{U}(n)\}$, there exists $n \in \mathbb{N}$ and open neighborhoods O, O' of $f(x_0), g(x_0)$ in Y , respectively, such that $\text{St}(O, \mathcal{U}(n)) \cap O' = \emptyset$. It is easy to check that

$$\text{St}(W(\{x_0\}; O), \mathcal{W}(n)) \cap W(\{x_0\}; O') = \emptyset.$$

Hence by the characterization, $C_k(X, Y)$ has a regular G_δ -diagonal. □

Corollary 2.5.7. *Let X be a compact space. If $\{\mathcal{U}(n) | n \in \mathbb{N}\}$ is a normal sequence of open covers of Y , then $\{\mathcal{W}(n) | n \in \mathbb{N}\}$, defined by the same method as (*) above, is also a normal sequence of open covers of $C_k(X, Y)$.*

Proof. We show $\mathcal{W}(n+1)^* < \mathcal{W}(n)$ under the condition $\mathcal{U}(n+1)^* < \mathcal{U}(n)$. Suppose $W(\delta) \cap W(\delta') \neq \emptyset$, $\delta, \delta' \in \Delta(n+1)$, where

$$\mathcal{K}(\delta) = \{K_1, \dots, K_s\}, \quad \mathcal{U}(\delta) = \{U_1, \dots, U_s\},$$

$$\mathcal{K}(\delta') = \{L_1, \dots, L_t\}, \quad \mathcal{U}(\delta') = \{V_1, \dots, V_t\},$$

$$\mathcal{U}(\delta) \cup \mathcal{U}(\delta') \subset \mathcal{U}(n+1).$$

For each i , take $U'_i \in \mathcal{U}(n)$ such that $\text{St}(U_i, \mathcal{U}(n+1)) \subset U'_i$. Let $\delta^* = (\mathcal{K}(\delta), \{U'_1, \dots, U'_s\}) \in \Delta(n)$. Then we can show $W(\delta') \subset W(\delta^*)$. Indeed, if $f \in W(\delta')$, then $f(L_j) \subset V_j$ for each $j = 1, \dots, t$. Since $\mathcal{K}(\delta')$ covers X , for each $K_i \in \mathcal{K}(\delta)$, let $N(i) = \{j | L_j \cap K_i \neq \emptyset, j = 1, \dots, t\}$, which implies

$$f(K_i) \subset \bigcup \{f(L_j) | j \in N(i)\} \subset \bigcup \{V_j | j \in N(i)\} \subset U'_i.$$

Therefore we have $f \in W(\delta^*)$. Hence we have $\mathcal{W}(n+1)^* < \mathcal{W}(n)$. \square

For each $\delta = (\mathcal{K}(\delta), \mathcal{U}(\delta)) \in \Delta(n)$, $n \in \mathbb{N}$, we define

$$W[\delta] = \bigcap \{ \langle X \times Y \setminus (K_i \times (Y \setminus U_i)) \rangle | K_i \in \mathcal{K}(\delta) \}.$$

Then obviously $W[\delta]$ is an open subset of $\mathcal{K}(X \times Y)$ such that $G(W(\delta)) = W[\delta] \cap G(C_k(X, Y))$. For each n , $\mathcal{W}[n] = \{W[\delta] | \delta \in \Delta(n)\}$ is an open cover of $G(C_k(X, Y))$ in $\mathcal{K}(X \times Y)$.

Lemma 2.5.8. *Let X be a compact space and let Y have a regular G_δ -diagonal. Then there exists a closed subspace \mathcal{K}_0 of $\mathcal{K}(X \times Y)$ containing $G(C_k(X, Y))$ such that if $f \in C(X, Y)$, $L \in \mathcal{K}_0$ with $L \neq G(f)$, then there exists $n \in \mathbb{N}$ such that $L \notin \text{St}(G(f), \mathcal{W}[n])^-$.*

Proof. Let

$$\mathcal{K}_0 = \{L \in \mathcal{K}(X \times Y) | \pi_X(L) = X\},$$

where $\pi_X : X \times Y \rightarrow X$ is the projection. As easily checked, \mathcal{K}_0 is a closed subspace of $\mathcal{K}(X \times Y)$. We show that $\mathcal{K}_0, \{\mathcal{W}[n]\}$ have the property. Let $G(f) \in$

$G(C_k(X, Y))$, $L \in \mathcal{K}_0$ with $G(f) \neq L$. Then there exists $\langle x, y \rangle \in L \setminus G(f)$. For the first case, suppose $y \notin f(X)$. Using compactness of $f(X)$ and the property of $\{\mathcal{U}(n)\}$, we can take $n \in \mathbb{N}$ and an open neighborhood O of y in Y such that $\text{St}(f(X), \mathcal{U}(n)) \cap O = \emptyset$. Then it is easy to see that $\langle X \times Y, X \times O \rangle$ is an open neighborhood of L in $\mathcal{K}(X \times Y)$ such that

$$\text{St}(G(f), \mathcal{W}[n]) \cap \langle X \times Y, X \times O \rangle = \emptyset.$$

For the second case, suppose $y \in f(X)$. Then $y \neq f(x)$. There exists disjoint open neighborhoods O, O' of $f(x), y$ in Y , respectively. We take the open neighborhood $\langle P_1 \times Q_1, P_2 \times Q_2 \rangle$ of $G(f)$ in $\mathcal{K}(X \times Y)$ as follows:

$$(1) \quad Q_1 = Y \setminus \{y\}, P_1 = f^{-1}(Q_1) \text{ and } P_2 \times Q_2 = f^{-1}(O') \times O'.$$

Since X is compact, there exists a closed cover $\{F_1, F_2\}$ of X such that $\emptyset \neq F_i \subset P_i$ for each $i = 1, 2$. By the property of $\{\mathcal{U}(n)\}$, for F_1 , there exists $n_0 \in \mathbb{N}$ and an open neighborhood $V(y)$ of y in Y such that

$$(2) \quad \text{St}(f(F_1), \mathcal{U}(n_0)) \cap V(y) = \emptyset.$$

By virtue of (1) and (2), we can easily show the following:

$$(3) \quad \text{If } G(f) \in W[\delta], \text{ where } \delta = (\mathcal{K}(\delta), \mathcal{U}(\delta)) \in \Delta(n_0), \mathcal{K}(\delta) = \{K_1, \dots, K_s\},$$

$$\mathcal{U}(\delta) = \{U_1, \dots, U_s\}, \text{ then for each } i = 1, \dots, s$$

$$(f^{-1}(O) \times V(y)) \cap (K_i \times U_i) = \emptyset.$$

From (3), it follows that $\langle f^{-1}(O) \times V(y), X \times Y \rangle$ is an open neighborhood of L in $\mathcal{K}(X \times Y)$ missing $\text{St}(G(f), \mathcal{W}[n_0])$. Hence we have $L \notin \text{St}(G(f), \mathcal{W}[n_0])^-$. \square

Lemma 2.5.9. *Let X' be a subspace of a $w\Delta$ -space X and suppose that there exists a sequence $\{\mathcal{U}(n) | n \in \mathbb{N}\}$ of open covers of X' in X such that for each $x \in X', y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that $y \notin \text{St}(x, \mathcal{U}(n))^-$. Then X' is a developable space.*

Proof. Let $\{\mathcal{U}'(n) | n \in \mathbb{N}\}$ be a $w\Delta$ -sequence for X . Let $\mathcal{V}(n) = (\mathcal{U}(n) \wedge \mathcal{U}'(n))|X'$, $n \in \mathbb{N}$. Without loss of generality, we can assume $\mathcal{V}(n+1) < \mathcal{V}(n)$ for each n . To see that $\{\mathcal{V}(n)\}$ forms a development for X' , let $p \in O \in \tau(X')$. Assume that for each n there exists $p_n \in \text{St}(p, \mathcal{V}(n)) \setminus O$. Then $\{p_n\}$ has a cluster point p' . But by the property of $\{\mathcal{U}(n)\}$, we have $p = p'$, a contradiction. Hence $\text{St}(p, \mathcal{V}(n)) \subset O$ for some n . \square

Theorem 2.5.10. *Let X be a hemicompact space. Then Y is a Moore space with a regular G_δ -diagonal if and only if so is $C_k(X, Y)$.*

Proof. If part follows easily from the fact that Moore spaces and regular G_δ -diagonals are hereditary and the fact $Y \leftrightarrow C_k(X, Y)$. Only if part: Since Moore spaces and regular G_δ -diagonals are countably productive and hereditary, by Lemma 2.5.5, it suffices to show it for a compact space X . Suppose that Y is a Moore space with a regular G_δ -diagonal and that X is a compact space. Since by Theorem 2.5.1 $C_k(X, Y)$ has a regular G_δ -diagonal, it suffices to show that $C_k(X, Y)$ is a Moore space. By Lemma 2.5.8, there exist a closed subspace \mathcal{K}_0 of $\mathcal{K}(X \times Y)$ containing $G(C_k(X, Y))$ and a sequence $\{\mathcal{W}[n] | n \in \mathbb{N}\}$ of open covers of $G(C_k(X, Y))$ in $\mathcal{K}(X \times Y)$ such that for each $G(f) \in G(C_k(X, Y))$, $L \in \mathcal{K}_0$ with $G(f) \neq L$, there exists $n \in \mathbb{N}$ such that $L \notin \text{St}(G(f), \mathcal{W}[n])^-$. Let $\mathcal{U}(n) = \mathcal{W}[n]| \mathcal{K}_0$, $n \in \mathbb{N}$. By [SM98, Theorem 2.2], there exists a perfect mapping of $\mathcal{K}(X \times Y)$ onto $\mathcal{K}(Y)$ because $\mathcal{K}(X)$ is compact. By [Miz95], $\mathcal{K}(Y)$ is a Moore space. Thus \mathcal{K}_0 is a $w\Delta$ -space. Using Lemma 2.5.9 with \mathcal{K}_0 and $\{\mathcal{U}(n)\}$, $X' = G(C_k(X, Y))$ (and hence $C_k(X, Y)$) is a Moore space. \square

Exercising our discussion used in the proof of Lemma 2.5.8, we can settle the following proposition, the result of which is well known as the Arens theorem [Are46, Theorem 7]. But this is the “topological” version of his proof.

Proposition 2.5.11. *If X is a compact space and Y is a metrizable space, then $C_k(X, Y)$ is metrizable.*

Proof. Let $\{\mathcal{U}(n)|n \in \mathbb{N}\}$ be a strong development for Y [Eng88, Theorem 5.4.2] such that $\mathcal{U}(n+1)^* < \mathcal{U}(n)$, $n \in \mathbb{N}$. Then by Corollary 2.5.7, $\mathcal{W}(n+1)^* < \mathcal{W}(n)$, $n \in \mathbb{N}$. So, for the metrizable of $C_k(X, Y)$ it suffices to show that $\{\mathcal{W}(n)\}$ is a development for $C_k(X, Y)$. Let $f \in W(K_1, \dots, K_s; O_1, \dots, O_s)$, where $K_i \in \mathcal{K}(X)$ and $O_i \in \tau(Y)$ for each i . Since $\{\mathcal{U}(n)\}$ is a strong development, for each i there exists $n(i) \in \mathbb{N}$ such that $\text{St}(f(K_i), \mathcal{U}(n(i))) \subset O_i$. Let $n = \max\{n(i)|i = 1, \dots, s\}$. For this n , we can easily show $\text{St}(f, \mathcal{W}(n)) \subset W(K_1, \dots, K_s; O_1, \dots, O_s)$. \square

2.6 On being σ -spaces of mapping spaces

We give an example of a σ -space X such that $C([0, 1], X)$ is not a σ -space, and show that if X is a compact metrizable space and the hyperspace $\mathcal{K}(Y)$ of Y is a σ -space, then $C(X, Y)$ is a σ -space.

2.6.1 Introduction

All spaces are assumed to be regular T_2 . \mathbb{N} always denotes all positive integers. For spaces X, Y let $C(X, Y)$ be the space of all continuous mappings of X into Y with compact open topology, the base for which consists of all subsets of the following form:

$$W(K_1, \dots, K_n; U_1, \dots, U_n) = \{f \in C(X, Y) \mid f(K_i) \subset U_i \text{ for each } i\},$$

where for each i , $K_i \in \mathcal{K}(X)$, the set of all non-empty compact subsets of X , $U_i \in \tau(Y)$, the topology of Y , and $n \in \mathbb{N}$. We give $\mathcal{K}(X)$ the finite topology, the base for which consists of all subsets of the following form:

$$\langle U_1, \dots, U_k \rangle = \left\{ K \in \mathcal{K}(X) \mid K \subset \bigcup_{i=1}^k U_i \text{ and } K \cap U_i \neq \emptyset \text{ for each } i \right\},$$

where $U_1, \dots, U_k \in \tau(X)$ and $k \in \mathbb{N}$.

In this section, we consider the heredity of the property of a space Y to $C(X, Y)$ when X is a compact metrizable space. Here, we give an example of a cosmic space X such that $C([0, 1], X)$ is not a σ -space. As a positive result, we settle that for a compact metrizable space X and a space Y with $\mathcal{K}(Y)$ a σ -space, $C(X, Y)$ is a σ -space.

As for undefined term, refer to [Gru84].

2.6.2 Results

Example 2.6.1. There exists a cosmic space X such that $C(I, X)$ is not a σ -space, where $I = [0, 1]$.

Construction. Let $X = \{(x, y) \mid y \geq 0\} \subset \mathbb{R}^2$, which is topologized by taking the neighborhood base $\mathcal{N}(p)$ for each point p in X as follows: For $p = (x, y)$, $y > 0$, $\mathcal{N}(p)$ is taken in the usual sense and for $p = (x, 0)$,

$$\mathcal{N}(p) = \{N(x; \varepsilon, \delta) \mid \varepsilon > 0, \delta > 1\},$$

where

$$N(x; \varepsilon, \delta) = \{p\} \cup \{(x', y') \in X \mid y' < \delta|x' - x|, |x' - x| < \varepsilon\}.$$

Then obviously X is a cosmic space. We show that $C(I, X)$ is not a σ -space. Assume that $C(I, X)$ has a net $\bigcup\{\mathcal{F}(n) \mid n \in \mathbb{N}\}$, where each $\mathcal{F}(n)$ is locally finite in $C(I, X)$. For each $x \in \mathbb{R}$, we define $f_x \in C(I, X)$ as follows:

$$f_x(t) = (t + x, t), \quad t \in I.$$

We fix ε_0, δ_0 such that $\varepsilon_0, \delta_0 > 1$. For each $x \in \mathbb{R}$, let

$$N(x) = N(x; \varepsilon_0, \delta_0).$$

Since $f_x \in W(I; N(x))$, there exists $F(x) \in \mathcal{F}(n(x))$ such that

$$f_x \in F(x) \subset W(I; N(x)).$$

There exists $n_0 \in \mathbb{N}$ such that

$$\text{Int}_{\mathbb{R}}(\text{Cl}_{\mathbb{R}}(\{x \mid n(x) = n_0\})) \neq \emptyset,$$

from which we choose r . Then there exists a sequence $\{x_m \mid m \in \mathbb{N}\}$ such that $\lim_{m \rightarrow \infty} x_m = r$ and for each m , $x_{m+1} < x_m$, $n(x_m) = n_0$ and $|r - x_1| < 1$. We can easily observe that $F(x_n) \neq F(x_m)$ if $n \neq m$. It is easy to check that any neighborhood of f_r intersects infinitely many members of $\{F(x_m) \mid m \in \mathbb{N}\}$. But this is a contradiction because $\{F(x_m)\}$ is locally finite in $C(I, X)$. Hence $C(I, X)$ is not a σ -space. \square

Lemma 2.6.2. *Let $n \in \mathbb{N}$. If X is a compact space and a space Y has a $G_\delta(n)$ -diagonal, then $C(X, Y)$ has a $G_\delta(n)$ -diagonal.*

Proof. Let $\{\mathcal{U}(n) \mid n \in \mathbb{N}\}$ be a sequence of open covers of Y such that $\{y\} = \bigcap \{S^n(y, \mathcal{U}(m)) \mid m \in \mathbb{N}\}$ for each $y \in Y$. For each $f \in C(X, Y)$ and $m \in \mathbb{N}$, there exists a finite cover $\mathcal{K}(f, m)$ of X such that $\mathcal{K}(f, m) \subset \mathcal{K}(X)$ and for each $K \in \mathcal{K}(f, m)$, $f(K) \subset U(K)$ for some $U(K) \in \mathcal{U}(m)$. Let

$$W(f, m) = \bigcap \{W(K; U(K)) \mid K \in \mathcal{K}(f, m)\}.$$

Then

$$\mathcal{W}(m) = \{W(f, m) \mid f \in C(X, Y)\}$$

is an open cover of $C(X, Y)$. We show that $\{\mathcal{W}(m) \mid m \in \mathbb{N}\}$ is a $G_\delta(n)$ -diagonal sequence for $C(X, Y)$. Let $f, g \in C(X, Y)$ with $f \neq g$. Then there exist $x \in X$ and $m \in \mathbb{N}$ such that $f(x) \notin S^n(g(x), \mathcal{U}(m))$. It is easy to check that $f \notin S^n(g, \mathcal{W}(m))$. This completes our task. \square

Remark 4. Taking [Gru84, Theorem 4.15] and the previous lemma with $m = 1$ into account, we can see that $C(I, X)$ in Example 2.6.1 is not a Σ -space in the sense of [Gru84, Definition 4.13].

In spite of the result in Corollary 2.3.8, we state the following with the direct proof:

Theorem 2.6.3. *Let X be a compact metrizable space and $\mathcal{K}(Y)$ a σ -space. Then $C(X, Y)$ is a σ -space.*

Proof. We note that $C(X, Y)$ can be embedded into $\mathcal{K}(X \times Y)$ by the embedding G such that

$$G(f) = \{(x, f(x)) \mid x \in X\}, \quad f \in C(X, Y).$$

Thus, it suffices to show that $G(C(X, Y))$ is a σ -space. Let \mathcal{B} be a countable closed net for X such that $X \in \mathcal{B}$. We define a subspace \mathcal{K}_0 of $\mathcal{K}(X \times Y)$ as

$$\mathcal{K}_0 = \{L \in \mathcal{K}(X \times Y) \mid \pi_X(L) = X\},$$

where for any closed subset C of X , $\pi_C : C \times Y \rightarrow C$ is the projection. It is easy to see that \mathcal{K}_0 is closed in $\mathcal{K}(X \times Y)$. By virtue of [SM98, Theorem 2.2] for each $B \in \mathcal{B}$ there exists a perfect mapping $\varphi_B : \mathcal{K}(B \times Y) \rightarrow \mathcal{K}(Y)$ such that $\varphi_B(K) = \pi'_B(K)$ for each $K \in \mathcal{K}(B \times Y)$, where $\pi'_B : B \times Y \rightarrow Y$ is the projection. Let $\psi_B : \mathcal{K}_0 \rightarrow \mathcal{K}(B \times Y)$ be a mapping such that $\psi_B(L) = L \cap (B \times Y)$ for each $L \in \mathcal{K}_0$. By assumption, there exists a σ -locally finite closed net \mathcal{F} for $\mathcal{K}(Y)$. Set

$$\mathcal{H}(B) = \psi_B^{-1} \varphi_B^{-1}(\mathcal{F}), \quad B \in \mathcal{B}$$

and

$$\mathcal{H} = \bigcup \{\mathcal{H}(B) \mid B \in \mathcal{B}\}.$$

Then obviously \mathcal{H} is a σ -locally finite family of closed subsets of \mathcal{K}_0 . Moreover, \mathcal{H} has the following property:

Claim: For each $f \in C(X, Y)$ and $L \in \mathcal{K}_0$ with $G(f) \neq L$, there exists $\hat{H} \in \mathcal{H}$ with $G(f) \in \hat{H}$ and a neighborhood \hat{W} of L in \mathcal{K}_0 such that $\hat{H} \cap \hat{W} = \emptyset$.

Proof of Claim: Suppose that we are given such $G(f)$ and L . Though $G(f) \neq L$ means a few cases, it suffices to show the existence of \hat{H} , \hat{W} for the case that there exists $(x, y) \in L \setminus G(f)$. Then there exists an open neighborhood $O_1 \times O_2$ of (x, y) in $X \times Y$ such that $(O_1 \times O_2) \cap G(f) = \emptyset$. Take $B \in \mathcal{B}$ such that $x \in B \subset O_1$. For this B , we easily have

$$\varphi_B(\psi_B(L)) \neq \varphi_B(\psi_B(G(f)))$$

in $\mathcal{K}(Y)$. Therefore there exist $\hat{F} \in \mathcal{F}$ with $\varphi_B(\psi_B(G(f))) \in \hat{F}$ and a neighborhood \hat{V} of $\varphi_B(\psi_B(L))$ in $\mathcal{K}(Y)$ such that $\hat{F} \cap \hat{V} = \emptyset$. Then it is easy to see that

$$\hat{W} = \psi_B^{-1} \varphi_B^{-1}(\hat{V}), \quad \hat{H} = \psi_B^{-1} \varphi_B^{-1}(\hat{F})$$

are the required sets.

Now, we show that $G(C(X, Y))$ has a σ -locally finite closed net. Let $\Delta\mathcal{H}$ be the totality of finite intersections of members of \mathcal{H} . Then it is also σ -locally finite in \mathcal{K}_0 . Set

$$\mathcal{P} = \Delta\mathcal{H} \mid G(C(X, Y)).$$

Then \mathcal{P} is a σ -locally finite family of closed subsets of $G(C(X, Y))$. We show that \mathcal{P} is a net for $G(C(X, Y))$. Let $G(f) \in \hat{O}$, where \hat{O} is open in \mathcal{K}_0 . If $\varphi_X^{-1}(\varphi_X(G(f))) \subset \hat{O}$, then there exists $\hat{F} \in \mathcal{F}$ such that

$$\varphi_X(G(f)) \in \hat{F} \subset \varphi_X^*(\hat{O}).$$

Hence we have

$$G(f) \in \varphi_X^{-1}(\hat{F}) \cap G(C(X, Y)) \subset \hat{O} \cap G(C(X, Y)),$$

and $\varphi_X^{-1}(\hat{F}) \cap G(C(X, Y)) \in \mathcal{P}$. Suppose $\varphi_X^{-1}(\varphi_X(G(f))) \not\subset \hat{O}$. By the claim, there exists $\hat{H} \in \Delta\mathcal{H}$ and a neighborhood \hat{W} of $\varphi_X^{-1}(\varphi_X(G(f))) \setminus \hat{O}$ such that $\hat{H} \cap \hat{W} = \emptyset$. There exists $\hat{F} \in \mathcal{F}$ such that

$$\varphi_X(G(f)) \in \hat{F} \subset \varphi_X^*(\hat{W} \cup \hat{O}).$$

Then we have

$$G(f) \in \varphi_X^{-1}(\hat{F}) \cap \hat{H} \cap G(C(X, Y)) \subset \hat{O} \cap G(C(X, Y)),$$

and $\varphi_X^{-1}(\hat{F}) \cap \hat{H} \cap G(C(X, Y)) \in \mathcal{P}$. This completes our task. \square

Corollary 2.6.4. *If X is a compact metrizable space and Y is a Lašnev space, then $C(X, Y)$ is a σ -space.*

Proof. By [Miz90, Theorem 4.12], if Y is a Lašnev space, then $\mathcal{K}(Y)$ is a σ -space. \square

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